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L-polytopes and equiangular lines

M. Deza[a, *], V.P. Grishukhin[b]

[a] CNRS, LIENS, Ecole Normale Superieure, 45, rue d'Ulm, 75230 Paris Cedex, France
[b] CEMI RAN, Russia

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Abstract

A construction providing sets of equiangular lines from integral lattices is given. Conditions are given when a set of equiangular lines with only one pillar determines an L-polytope of an integral lattice. Several examples of L-polytopes related to sets of equiangular lines are given.

Keywords: Integral lattice; Equiangular lines; Polytope

0. Introduction

We continue (see [7, 8]) to study relations between L-polytopes and combinatorial objects such as distance-regular graphs, two-graphs, sets of lines with small number of angles, and others.

In fact, in this work we study sets of equiangular lines. We show that an integral lattice naturally generates sets of equiangular lines.

If we take the intersection point of a set of equiangular lines as the center of a sphere, then the intersection points of the lines with the sphere form a spherical distance space with two distances between nonantipodal points. Edges of the graph corresponding to the set of equiangular lines join pairs of nonantipodal points with largest distance. The vectors spanning the lines provide a spherical representation of the graph.

We are interested when the distance space is hypermetric. In other words, when the convex hull of points of the spherical distance space is an L-polytope of a lattice affinely generated by the points. If the distance space is hypermetric we call the corresponding spherical representation of the graph *hypermetric* representation.

In [7] we studied hypermetric graphs, i.e. graphs whose path metric is hypermetric. Each hypermetric graph can be represented by points on a sphere such that the path distance between vertices is equal to the squared Euclidean distance between the points. Radius vectors of the points define a hypermetric representation of the graph.

* Corresponding author.

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In particular, we studied hypermetric distance-regular graphs. It is well known that every distance-regular graph has a spherical representation related to each eigenvalue of its adjacency matrix. We show in [7] that for a class of distance-regular graphs the representation related to the second largest eigenvalue is hypermetric representation of path metric. In particular, the class contains Taylor graphs related to regular two-graphs with smallest eigenvalue -3.

There is correspondence between Taylor graphs, regular two-graphs and special sets of equiangular lines. A Taylor graph G has 4 eigenvalues: 2 eigenvalues, k, the degree of G, and -1, and 2 eigenvalues which are minus eigenvalues of the ± 1 adjacency matrix of the corresponding two-graph. The second largest eigenvalue of G is minus the smallest eigenvalue of the two-graph. It is shown in Section 8 that the vectors of the above-mentioned representations of the distance-regular graph G related to the eigenvalues distinct from k and -1 span two distinct sets of equiangular lines. We conjecture that the representations are hypermetric if the dimension is sufficiently large.

We show in [8] that the famous set of 276 equiangular lines, corresponding to the unique regular two-graph on 276 points, provides an extreme hypermetric distance space. In other words, the representation of the corresponding Taylor graph related to the second largest eigenvalue (equal to 5) is hypermetric.

In Section 1 we consider an L-polytope P which is a convex hull of endpoints of minimal vectors of a lattice L lying in the affine hyperplane $2cx = c^2$, where c is a vector of L. The polytope P is symmetric, and pairs of its opposite vertices span lines with small number of angles between them. The construction of Section 1 is a generalization of one of [13].

In Section 2 we study a set of vectors spanning the lines. We call it (M, k)-*system* if M is the norm (= squared length) of vectors, and an inner product of 2 vectors of the system is integral, not greater than k, and has the same parity for all pairs of vectors. An $(M, 1)$-system for even (odd) M is a frame (spans a set of equiangular lines, respectively). We show in Section 3 that a $(3, 1)$-system is related to a root lattice. This relation allows a complete classification of maximal $(3, 1)$-systems.

In Sections 4-9 we study $(2t+1, 1)$-systems spanning sets of equiangular lines at angle $\arccos 1/(2t+1)$. In Section 4 we define a pillar $(2t+1, 1)$-system as a set $\mathbb{P}$ of equiangular vectors of the norm $2t+1$ such that there is a vector e of norm 1 with $ve = \pm 1$ for all $v \in \mathbb{P}$. The definition generalizes the definition of a pillar of Lemmens and Seidel [10]. We study pillar sets in detail in Section 5. In fact pillar $(2t+1, 1)$-systems are determined by graphs with largest eigenvalue not greater than t. In Section 6 we study corresponding L-polytopes. We show here that not all maximal $(2t+1, 1)$-systems determine L-polytopes. This fact clarifies a question of [13]. Note that all L-polytopes of the Leech lattice relate to maximal pillar $(5, 1)$-systems of dimension 24.

We show in Section 7 that there are restrictions on graphs corresponding pillars if there are more than one pillar. Here we introduce an extreme $(2t+1)$-system as a system satisfying simultaneously the special and absolute bounds.

Section 8 describes exact relation between a Taylor graph and a pair of complementary regular two-graphs.

In Section 9 we give some examples of L-polytopes related to $(2t+1,1)$-systems with several pillars for $t = 1, 2$.

1. L-polytopes related to minimal vectors of a lattice

We give simple facts which imply that integral lattices furnish systems of lines with small numbers of values of angles between the lines.

Recall that a discrete additive subgroup L of $\mathbb{R}^k$ is called a *lattice*. A subset L' of $\mathbb{R}^k$ is called an *affine lattice* if L' is the image of a lattice by some translation, i.e. $L' = L + b = \{a + b\colon a \in L\}$ for some lattice L and some $b \in \mathbb{R}^k$, $b \neq 0$. The rank of the subgroup L is called *dimension* of L. A difference of 2 points of an affine lattice is called a *lattice vector*.

For a finite set X we denote by $|X|$ its cardinality. The squared length of a vector a, a^2, is called the *norm* of the vector a. Minimal norm of lattice vectors of an (affine) lattice L is called the *minimal norm* of L.

Let L be a k-dimensional lattice and let S be a sphere in $\mathbb{R}^k$ with center c and radius r, $S = \{x \in \mathbb{R}^k\colon (x-c)^2 = r^2\}$. Then S is called an *empty sphere* (or *hole*) in L if $(v-c)^2 \geqslant r^2$ holds for all $v \in L$ and the set $S \cap L$ has rank k (i.e. affine rank $k+1$), i.e. no lattice point lies in the interior of the ball with boundary sphere S but S is completely determined by the lattice points lying on it. The convex hull of $S \cap L$, $P = \operatorname{conv}(S \cap L)$, is called an *L-polytope* (a *Delone polytope*) of the lattice L.

Let L be a lattice and let $\mathbb{M}$ be the set of its minimal vectors. Let m be the minimal norm of the lattice L. For $c \in L$, $c \neq 0$ we set

$$\mathbb{A}(c) = \{a \in \mathbb{M}\colon 2ac = c^2\}.$$

Proposition 1.1. *If* $|\mathbb{A}(c)| > 1$, $c \neq 0$, *then the convex hull* $P(c)$ *of endpoints of all* $a \in \mathbb{A}(c)$ *is an L-polytope.*

Proof. The module $L(c) = \{\sum z_a a\colon a \in \mathbb{A}(c), \sum z_a = 1, z_a \in \mathbb{Z}\}$ is a sublattice of L. An affine image of $L(c)$ is the section of the lattice L by the hyperplane $H = \{x\colon 2xc = c^2\}$. We show that $P(c)$ is an L-polytope of $L(c)$. In fact, $P(c)$ is inscribed into the sphere $S(c)$ which is an intersection of the sphere $x^2 = m$ and the hyperplane H. Since $L(c)$ is a sublattice of L, $b^2 \geqslant m$ for all $b \in L(c)$. Hence no point of $L(c)$ lies inside the sphere $x^2 = m$, that is the sphere $S(c)$ is empty and $P(c)$ is an L-polytopes of the lattice $L(c)$. □

Now we describe some simple properties of $\mathbb{A}(c)$.

Proposition 1.2. (1) $\mathbb{A}(c) \neq \emptyset$ *if and only if* $c = a + b$ *for some* $a, b \in \mathbb{M}$. *Then* $a, b \in \mathbb{A}(c)$.
(2) *If* $\mathbb{A}(c) \neq \emptyset$, *then* $c - a \in \mathbb{A}(c)$ *for all* $a \in \mathbb{A}(c)$.

(3) *If* $\mathbb{A}(c) \neq \emptyset$, *then* $|\mathbb{A}(c)| = 1$ *if and only if* $c^2 = 4m$.
(4) *If* $|\mathbb{A}(c)| > 1$, *then for* $a, b \in \mathbb{A}(c)$, $a \neq b$,

$$\begin{aligned} &\text{either } ab = c^2/2 - m \quad \text{and} \quad b = c - a, \\ &\text{or } (c^2 - m)/2 \leqslant ab \leqslant m/2 \quad \text{and} \quad b \neq c - a. \end{aligned} \tag{1.1}$$

(5) *If* $|\mathbb{A}(c)| > 2$, *then* $m \leqslant c^2 \leqslant 2m$.
(6) *If* $2m < c^2 < 4m$, *then* $|\mathbb{A}(c)| = 2$.
(7) *If* $\mathbb{A}(c) \neq \emptyset$ *and* L *is an integral lattice, then* c^2 *is an even integer.*

Proof. (1) Let $a, b \in \mathbb{M}$. We set $c = a + b$. Since $a^2 = b^2 = m$, $c^2 = 2a^2 + 2ab = 2a(a + b) = 2ac$. Hence $a \in \mathbb{A}(c)$ and $\mathbb{A}(c) \neq \emptyset$. Similarly $b \in \mathbb{A}(c)$. Suppose that $\mathbb{A}(c) \neq \emptyset$. Let $a \in \mathbb{A}(c)$. We set $b = c - a$. Obviously $b \in L$. Now the equality $2ac = c^2$ can be written as $(c - a)^2 = a^2 = m$, that is $b \in \mathbb{M}$ and $c = a + b$ for $a, b \in \mathbb{M}$.

(2) It was just before shown that $c - a \in \mathbb{M}$ for $a \in \mathbb{A}(c)$. The equality $2(c - a)c = 2c^2 - 2ac = c^2$ implies $c - a \in \mathbb{A}(c)$.

(3) Let $\mathbb{A}(c) = \{a\}$. According to (2), $c - a \in \mathbb{A}(c)$. Hence $c - a = a$, i.e. $c = 2a$ and $c^2 = 4m$. Now let $c^2 = 4m$ and let $a \in \mathbb{A}(c)$. By (2), $b = c - a \in \mathbb{A}(c)$. We have $4m = c^2 = (a + b)^2 = 2m + 2ab$. Hence $ab = m$, which is possible if and only if $a = b$. Hence $c = 2a$ for all $a \in \mathbb{A}(c)$, since a is an arbitrary element of $\mathbb{A}(c)$, and $\mathbb{A}(c) = \{a\}$.

(4) Let $a, b \in \mathbb{A}(c)$ and $a \neq b$. Since $a - b \in L$, $(a - b)^2 \geqslant m$. The inequality implies $ab \leqslant m/2$, the right-hand side of (1.1). Since $a^2 = b^2 = m$ and $2ac = 2bc = c^2$, $(a + b - c)^2 = 2m - c^2 + 2ab$. Hence,

if $a + b - c = 0$, then $ab = c^2/2 - m$ and $b = c - a$,

if $a + b - c \neq 0$, then $(a + b - c)^2 \geqslant m$. This implies $ab \geqslant (c^2 - m)/2$, the left-hand side of (1.1).

(5) if $|\mathbb{A}(c)| > 2$, then there are $a, b \in \mathbb{A}(c)$ such that $b \neq c - a$. Hence ab satisfies (1.1). The inequalities of (1.1) imply $c^2 \leqslant 2m$. Since $c \in L$, $c \neq 0$, and m is minimal norm of L, $c^2 \geqslant m$.

(6) If $2m < c^2 < 4m$, then by (1.1) there is no pair of vectors $a, b \in \mathbb{A}(c)$ such that $a + b \neq c$. Hence $\mathbb{A}(c) = \{a, c - a\}$.

(7) If L is integral, then m and ab are integers. Hence $c^2 = (a + b)^2 = 2m + 2ab$ is an even integer. □

Now we take the center $c/2$ of the L-polytope $P(c)$ as the new origin and consider the set of vectors

$$\mathbb{V}(c) = \{u\colon u = 2^{1/2}(a - c/2),\ a \in \mathbb{A}(c)\}.$$

We call a set $\mathbb{X}$ of vectors *symmetric* if $x \in \mathbb{X}$ implies $-x \in \mathbb{X}$.

Proposition 1.3. (1) *$\mathbb{V}(c)$ is a symmetric set of vectors of the norm $2m - c^2/2$ and $uc = 0$ for all $u \in \mathbb{V}(c)$.*
(2) *For $u, v \in \mathbb{V}(c)$, $v \neq \pm u$,*

$$-(m - c^2/2) \leqslant uv \leqslant (m - c^2/2). \tag{1.2}$$

(3) *If L is integral, then inner products uv and norms u^2 are integers of the same parity for all $u, v \in \mathbb{V}(c)$.*
(4) *The convex hull of endpoints of all $u \in \mathbb{V}(c)$ is the L-polytope $2^{1/2}P(c)$.*

Proof. (1) Let $u \in \mathbb{V}(c)$. Then $uc = 2^{1/2}(ac - c^2/2) = 0$ and $u^2 = 2(m - c^2/4)$. Let $b = c - a \in \mathbb{A}(c)$. Then $2^{1/2}(b - c/2) = 2^{1/2}(c/2 - a) = -u$ belongs to $\mathbb{V}(c)$.

(2) For $u = 2^{1/2}(a - c/2)$ and $v = 2^{1/2}(b - c/2)$ we obtain

$$uv = 2ab - c^2/2. \tag{1.3}$$

If $ab = c^2/2 - m$, then $b = c - a$ by Proposition 1.2(4) and consequently $v = -u$. Hence if $v \neq \pm u$, ab satisfies (1.1). The inequalities (1.1) and the equality (1.3) implies (1.2).

(3) If L is integral, then ab is an integer for all $a, b \in L$. By Proposition 1.2(7), the norm c^2 of the vector c is an even integer. Now the equality (1.3) shows that inner products uv and the norms u^2 are integers and the integers have the same parity for all $u, v \in \mathbb{V}(c)$, namely, the parity of $C^2/2$.

(4) Obviously the convex hull of endpoints of $u \in \mathbb{V}(c)$ is $2^{1/2}P(c)$, the homothety of $P(c)$. But a homothety of an L-polytope is an L-polytope. □

Now we consider special cases of the systems $\mathbb{V}(c)$. Of course, the nontrivial case $m \leqslant c^2 \leqslant 2m$ is of the most interest. Recall that a symmetric set with mutually orthogonal nonopposite vectors of equal norms is called a *frame*.

Consider at first the case $c^2 = 2m$.

Proposition 1.4. *If $c^2 = 2m$, then $\mathbb{V}(c)$ is a frame and $|\mathbb{V}(c)| = 2\dim\mathbb{V}(c)$.*

Proof. In the case the inequalities of (1.2) take the form of the equality $uv = 0$ for nonopposite vectors u, v. Hence $\mathbb{V}(c)$ is a frame. Since any frame contains a basis of the space spanned by it, the frame has $2\dim|\mathbb{V}(c)|$ vectors, $|\mathbb{V}(c)| = 2\dim\mathbb{V}(c)$. □

From now on we suppose that L is an integral lattice and $c^2 < 2m$. Consider the value of c^2 next to the value $2m$, i.e. $c^2 = 2m - 2$ (recall that, by Proposition 1.2(7), c^2 is an even integer).

Proposition 1.5. *Let $m \geqslant 2$, and $c^2 = 2m - 2$. If m is even, then $\mathbb{V}(c)$ spans a set of equiangular lines at angle* arccos $1/(m+1)$. *If m is odd, then $\mathbb{V}(c)$ is a frame.*

Proof. In the case the inequalities of (1.2) and the equality (1.3) take the form

$$-1 \leqslant uv \leqslant 1, \quad uv = 2ab - m + 1.$$

If m is odd, then uv is an even integer. Since $|uv| \leqslant 1$, $uv = 0$. Hence $\mathbb{V}(c)$ is a frame and $\mathbb{V}(c)$ spans a set of equiangular lines at angle $\pi/2$.

If m is even, then uv is an odd integer. Hence $uv = \pm 1$, i.e. there is only one nonobtuse angle between nonopposite vectors. Hence the vectors of $\mathbb{V}(c)$ span equiangular lines. Since the norm of vectors of $\mathbb{V}(c)$ is $m + 1$, the nonobtuse angle is equal to $\arccos 1/(m + 1)$, where $m + 1$ is odd. □

Proposition 1.6. *Let $m \geqslant 4$, and $c^2 = 2m - 4$. If m is odd, then $\mathbb{V}(c)$ spans a set of equiangular lines at angle* $\arccos 1/(m + 2)$. *If m is even, then $\mathbb{V}(c)$ spans a set of lines at 2 angles one of which is $\pi/2$.*

Proof. In the case (1.2) and (1.3) take the form

$$-2 \leqslant uv \leqslant 2, \quad uv = 2ab - m + 2.$$

The norm of vectors of $\mathbb{V}(c)$ is equal to $v^2 = m + 2$. If m is odd, then uv is odd, too. It follows that $uv = \pm 1$ and we obtain a set of equiangular lines at angle $\arccos 1/(m + 2)$ spanned by vectors of odd norm $m + 2$.

If m is even, then uv is even and $uv = 0, \pm 2$. If $uv = 0$, then u and v are orthogonal. Otherwise the obtuse angle between the lines spanned by u and v is $\arccos 2/(m + 2)$. □

2. (M, k)-systems

We want to describe abstractly the sets $\mathbb{V}(c)$. We denote by $M = 2m - c^2/2$ the norm of vectors of $\mathbb{V}(c)$. We know that $\mathbb{V}(c)$ is nontrivial if $m \leqslant c^2 \leqslant 2m$. Hence the norm of vectors of nontrivial $\mathbb{V}(c)$ satisfies the inequalities

$$m \leqslant M \leqslant 3m/2.$$

Since c^2 is an even integer, we can set

$$c^2 = 2(m - k),$$

where k is an integer and $0 \leqslant k \leqslant m/2$. In this notations the inequalities (1.2) take the form

$$-k \leqslant uv \leqslant k. \tag{2.1}$$

We have $M = m + k$, and M and uv have the same parity. If $k = [m/2]$, then $M = [3m/2]$, i.e. $m \approx 2M/3$ and

$$k \leqslant M/3. \tag{2.2}$$

We say that a set of vectors satisfies a *parity condition* if inner products and norms of vectors of the set are integers of the same parity.

Note that if M and k are even, then uv are even, too, and the scaled system $2^{-1/2}\mathbb{V}(c)$ consists of the vectors of integral norm with integral inner products satisfying (2.1) and (2.2).

Definition 2.1. A symmetric set $\mathbb{V}$ of vectors of an integral norm M is called *(M,k)-system* if $\mathbb{V}$ satisfies the parity condition and for any $u, v \in \mathbb{V}$, $u \neq \pm v$, the inner product uv is an integer satisfying (2.1) with $k \leqslant M/3$ and there exist $u, v \in \mathbb{V}$ with $uv = k$.

An (M,k)-system is called *irreducible* if it cannot be partitioned into two (M,k)-systems such that any vector of one system is orthogonal to any vector of other one. A *dimension* of an (M,k)-system $\mathbb{V}$, $\dim \mathbb{V}$, is the dimension of the space spanned by $\mathbb{V}$. An (M,k)-system is called *maximal* if it cannot be enlarged without increasing the dimension of the (M,k)-system. An (M,k)-system is called *extremal* if it has maximal cardinality among all (M,k)-systems of the same dimension. A *size* of an (M,k)-system $\mathbb{V}$, $v(\mathbb{V})$, is the cardinality of maximal subset of $\mathbb{V}$ containing no pair of opposite vectors.

Propositions 1.4 and 1.5 give us the following two examples.

Example 2.2. An $(M,0)$-system is a frame.

Example 2.3. For M odd, vectors of an $(M,1)$-system span equiangular lines at angle $\arccos 1/M$.

Counterexample 2.4. A root system is not a $(2,k)$-system, since there are pairs of nonorthogonal roots with inner product $1 > \frac{2}{3}$.

Counterexample 2.5. Sets of minimal vectors of many integral lattices of minimal norm 4 are not $(4,k)$-systems, since inner product can take the value $2 > \frac{4}{3}$.

Proposition 2.6. *The $\mathbb{Z}$-module $L'(\mathbb{V})$ affinely generated by an (M,k)-system $\mathbb{V}$ is an affine integral lattice.*

Proof. Let $w = \{\sum z_u u\colon \sum z_u = 1, u \in \mathbb{V}\}$ be an element of $L'(\mathbb{V})$. Since inner products of vectors of $\mathbb{V}$ are integral, the norm of w is an integer. Hence $L'(\mathbb{V})$ is discrete and consequently it is a lattice. The lattice is integral, since inner products of vectors of $\mathbb{V}$ are integral. The lattice is affine, since the origin 0 is not a lattice point. $\square$

We set $L_0(\mathbb{V}) = 2^{-1/2}L'(\mathbb{V})$.

Proposition 2.7. *The lattice $L_0(\mathbb{V})$ is integral and even. The minimal norm of $L_0(\mathbb{V})$ is $\leqslant M - k$.*

Proof. Consider at first a lattice vector $2^{-1/2}(u-v)$ for $u, v \in \mathbb{V}$. The norm $(u-v)^2/2 = M - uv$ is integral and divisible by 2, since M and uv have the same parity. Note that the parity condition implies that the inner product $(u-u')(v-v')$ is even for $u, u', v, v' \in \mathbb{V}$. Any lattice vector of $L'(\mathbb{V})$ can be written as $v - u_0 = \sum z_u u - u_0 = \sum z_u(u-u_0)$ for fixed $u_0 \in \mathbb{V}$. Hence $(v-u_0)^2/2 = \sum z_u^2(u-u_0)^2/2 + \sum z_u z_v(u-u_0)(v-u_0)$ is even. The minimal norm is not greater than $M-k$, since for $u, v \in \mathbb{V}$, $uv = k$, $(u-v)^2/2 = M-k$. □

Proposition 1.3(4) justifies introducing the following

Conjecture 2.8. The convex hull $P(\mathbb{V})$ of endpoints of a maximal (M,k)-system $\mathbb{V}$ is an L-polytope of the lattice $L_0(\mathbb{V})$.

We show in Section 6 that Conjecture 2.8 is not true in such general wording.

We can restate Conjecture 2.8 in another form. Let $\dim L_0(\mathbb{V}) = d$. We embed the lattice $L_0(\mathbb{V})$ in a $(d+1)$-dimensional lattice $L(\mathbb{V})$. The construction is called a *superposition of layers*. We represent the lattice $L_0(\mathbb{V})$ of dimension $d = \dim \mathbb{V}$ as a layer of a lattice of dimension $d+1$. Suppose that $\mathbb{V}$ satisfies the parity condition. Represent the space spanned $\mathbb{V}$ as a hyperplane of $\mathbb{R}^{d+1}$. Let c be a vector orthogonal to the hyperplane. Let the norm of c be $c^2 = 2M - 4k$. We relate to each vector $u \in \mathbb{V}$ the vector

$$a_u = 2^{-1/2}u + c/2.$$

Let $L(\mathbb{V})$ be the $\mathbb{Z}$-module generated by the set $A(\mathbb{V}) = \{a_u\colon u \in \mathbb{V}\}$. All the vectors of $A(\mathbb{V})$ have the same integral norm $a_u^2 = M - k$. The inner products

$$a_u a_v = (uv + M - 2k)/2$$

are integral, too, since $\mathbb{V}$ satisfies the parity condition. Now, the proof similar to the proof of Proposition 2.6 shows that $L(\mathbb{V})$ is an integral lattice.

Since $a_u - a_v = 2^{-1/2}(u-v)$, the lattice $L(\mathbb{V})$ is generated by an arbitrary vector a_u and by lattice vectors of $L_0(\mathbb{V})$. This means that $L(\mathbb{V})$ is, in fact, composed by layers isomorphic to the lattice $L_0(\mathbb{V})$.

The main question here is whether the minimal norm of the lattice $L(\mathbb{V})$ is equal to $M-k$, the norm of the vectors of $\mathbb{A}(\mathbb{V})$. If so, $\mathbb{V}$ can be obtained from the lattice $L(\mathbb{V})$ by the construction of Section 1. We note that $c \in L(\mathbb{V})$, since $c = a_u + a_{-u}$.

Recall that $P(\mathbb{V})$ is the convex hull of endpoints of the vectors of $\mathbb{V}$.

Proposition 2.9. *If the minimal norm of $L_0(\mathbb{V})$ is equal to $M-k$, then $P(\mathbb{V})$ is an L-polytope of the lattice $L_0(\mathbb{V})$ if and only if the minimal norm of the lattice $L(\mathbb{V})$ is equal to $M-k$.*

Proof. If minimal norm of $L(\mathbb{V})$ is equal to $M-k$, then we can apply the proof of Proposition 1.1. Let $P(\mathbb{V})$ be an L-polytope of $L_0(\mathbb{V})$. Minimal vectors of $L(\mathbb{V})$ are minimal vectors of $L_0(\mathbb{V})$ and vectors with endpoints in the layer $L_0(\mathbb{V})$. If a vector of the second type has norm less than $M - k$, then its endpoint lies inside the L-polytope $P(\mathbb{V})$, a contradiction. □

Therefore, if the minimal norm of $L_0(\mathbb{V})$ is $M - k$, then Proposition 2.9 proves the equivalence of Conjecture 2.8 to the following.

Conjecture 2.10. If $\mathbb{V}$ is a maximal (M, k)-system, then minimal norm of the lattice $L(\mathbb{V})$ is $M - k$.

Conjecture 2.10 is a wide generalization of a question of Neumaier [13].

Note that endpoints of vectors of an (M, k)-system $\mathbb{V}$ compose a spherical distance space $(\mathbb{V}, d_0)$. The distance $d_0(u, v)$ and the inner product uv are related as follows:

$$d_0(u, v) = (u - v)^2 = 2(M - uv).$$

Recall that any subset of vertices of an L-polytope compose a hypermetric space, and any hypermetric space can be realized as a subset of vertices of an L-polytope [8]. Hence Conjecture 2.8 (and consequently Conjecture 2.10) imply the following.

Conjecture 2.11. If $\mathbb{V}$ is a maximal (M, k)-system, then the distance space $(\mathbb{V}, d_0)$ is a hypermetric distance space.

Recall that any $(M, 0)$-system is a frame. So is any $(M, 1)$-system with an even M. An $(M, 1)$-system with an odd M spans a set of equiangular lines. Hence the above conjectures are true for any $(M, 0)$-system $\mathbb{V}$. The corresponding L-polytope is the cross polytope. The lattice $L_0(\mathbb{V})$ is D_n, $n = \dim \mathbb{V}$, and $L(\mathbb{V})$ is A_{n+1}.

3. (3, 1)-systems or equiangular lines at angle arccos 1/3

Now we prove the validity of the conjectures for (3, 1)-systems.

For a (3, 1)-system $\mathbb{V}$ the lattice $L(\mathbb{V})$ is generated by the vectors of the norm $M - k = 3 - 1 = 2$ with inner products 0 and 1. In this case the vector c has the norm 2 too and $(a, c) = 1$ for all $a \in \mathbb{A}(\mathbb{V})$.

Proposition 3.1. *For a* (3, 1)*-system* $\mathbb{V}$ *the distance space* $(\mathbb{V}, d_0)$ *is hypermetric. If* $\mathbb{V}$ *is maximal, then the polytope* $P(\mathbb{V})$ *is an L-polytope.*

Proof. In this case the lattice $L(\mathbb{V})$ is generated by vectors of the norm 2 with integral inner products. The norm of any vector $x = \sum z_a a \in L(\mathbb{V})$, $x^2 = \sum z_a^2 a^2 + 2\sum_{a \neq b} z_a z_b ab = 2(\sum z_a^2 + \sum z_a z_b ab)$, is even. Hence the minimal norm of $L(\mathbb{V})$ is 2, the norm of vectors of $\mathbb{A}(\mathbb{V})$, and Conjectures 2.10, 2.8 and 2.11 are true. □

Remark 3.2. In this case the lattice $L(\mathbb{V})$ is a root lattice. Hence all maximal (3, 1)-systems can be classified.

Table 1

$L(\mathbb{V})$	A_n	D_n	E_6	E_7	E_8
$L_0(\mathbb{V})$	D_{n-1}	$A_1 \otimes D_{n-2}$	A_5	D_6	E_7
$P(\mathbb{V})$	$CP(n-1)$	$I \otimes CP(n-2)$	$J(6,3)$	Half cube	Gosset polytope
$v(\mathbb{V})$	$n-1$	$2(n-2)$	10	16	28

Recall the root lattices are $A_n (n \geqslant 1)$, $D_n (n \geqslant 2)$, E_6, E_7, E_8. There are some isomorphisms for small n: $A_1 \cong \mathbb{Z}^1$, $D_2 \cong \mathbb{Z}^2$, $A_3 \cong D_3$. In Table 1 we give (3, 1)-systems corresponding to the root lattices. There $CP(n)$ is a cross polytope of dimension n, $CP(n) \otimes I$ is the direct product of a segment I of the length 2 and $CP(n)$, $J(6,3)$ is a 5-dimensional Johnson polytope. In other words $J(6,3)$ is the convex hull of vertices of the middle 3-layer of 6-dimensional cube. We note that $L_0(\mathbb{V})$ for $\mathbb{V}$ of types A_n and D_n are not root lattices, since each its basis contains a vector of norm at least 4.

The classification of extremal (3, 1)-systems was given in [17] by the method that we use in sections below. Theorem 1 of [12] shows that maximal (3, 1)-systems coincide with closed (1/3)-systems of Neumaier.

4. Pillars of equiangular lines

Consider an $(M, 1)$-system $\mathbb{V}$ with $M = 2t + 1$, an odd integer. Since vv' for $v, v' \in \mathbb{V}$, $v \neq v'$, take only 2 values ± 1, the $(2t+1, 1)$-system is related to a set of equiangular lines at angle arccos $1/(2t+1)$, where t is a natural number. The set $\mathbb{V}$ is a symmetric set of vectors of norm $2t+1$ spanning the lines.

It is well known [3, 10, 14, 20] that any set of equiangular lines corresponds to a 2-graph and to a switching class of graphs. Let $\mathbb{V}_1$ be a subset of $\mathbb{V}$ containing just one vector from each pair of opposite vectors. Let $G(\mathbb{V}_1)$ be a graph with the set of vertices $\mathbb{V}_1$. Vertices v and v' are adjacent in $G(\mathbb{V}_1)$ if and only if $vv' = -1$. If we change $v \in \mathbb{V}_1$ by its opposite $-v$, then the new graph $G(\mathbb{V}_1 - \{v\} + \{-v\})$ is obtained from $G(\mathbb{V}_1)$ by switching the vertex v.

Let $\mathscr{G}(\mathbb{V})$ be the switching class of graphs $G(\mathbb{V}_1)$ for $\mathbb{V}_1 \subseteq \mathbb{V}$. Obviously any graph of the class $\mathscr{G}(\mathbb{V})$ has the form $G(\mathbb{V}_1)$ for some $\mathbb{V}_1 \subseteq \mathbb{V}$ without pairs of opposite vectors.

Let $G(V, E)$ be a graph with vertex set V and edges set E. Recall [3, 14] that a map $v \to p_v$, where $p_v \in \mathbb{R}^n$, is called a (p, q, r)-representation of G if

$$p_v^2 = p, \ \ p_v p_u = q \quad \text{if } (v,u) \in E, \quad p_v p_u = r \quad \text{if } (v,u) \notin E.$$

Obviously the map $v \to v$ is a $(2t+1, -1, 1)$-representation of the graph $G(\mathbb{V}_1)$. The Gram matrix of vectors $v \in \mathbb{V}_1$ has the form

$$(2t+1)I + J - I - 2A = 2(tI - A) + J, \tag{4.1}$$

where A is adjacency matrix of the graph $G(\mathbb{V}_1)$, I is an identity matrix and J is a matrix all of whose elements are 1. This Gram matrix makes obvious the following

Proposition 4.1 (Brouwer et al. [3, Proposition 3.8.1]). *If a graph G is switching equivalent to a graph with largest eigenvalue $\leqslant t$, then G has a $(2t+1, -1, 1)$-representation.*

Proof. Let G' be a graph with largest eigenvalue $\lambda_{\max}(G') \leqslant t$ in the switching class of G. Since $\lambda_{\max}(G') \leqslant t$, the matrix $2(tI - A)$ is positive semidefinite, i.e. the matrix is the Gram matrix of vectors x'_v of norm $2t$. Let e' be a vector of norm 1 which is orthogonal to the space spanned by all x'_v. It is easy to verify that the Gram matrix of vectors $v' = x'_v + e'$ is the matrix (4.1), i.e. the vectors give a $(2t+1, -1, 1)$-representation of G'. We obtain a $(2t+1, -1, 1)$-representation of G if we change in the representation of G' the sign of vectors v' corresponding to switched vertices. □

For any set $\mathbb{X}$ of vectors we set

$$a(\mathbb{X}) = \sum\{v\colon v \in \mathbb{X}\}. \tag{4.2}$$

Consider subsets of vectors of $\mathbb{V}$ with mutual inner products -1.

Proposition 4.2 (Lemmens and Seidel [10] and Neumaier [14]). *A maximal subset $\mathbb{K} \subseteq \mathbb{V}$ of vectors with mutual inner products -1 contains at most $2t+2$ vectors. If $|\mathbb{K}| = 2t+2$, then $a(\mathbb{K}) = 0$.*

Proof. Set $k = |\mathbb{K}|$. Recall that $v^2 = 2t+1$ for all $v \in \mathbb{V}$. Hence we have $0 \leqslant a(\mathbb{K})^2 = \sum\{v^2\colon\ v \in \mathbb{K}\} + 2\sum\{vv'\colon\ v, v' \in \mathbb{K}\} = (2t+1)k + k(k-1)(-1) = k(2t+1-k+1)$, i.e. $k \leqslant 2t+2$, and if $k = 2t+2$, then $a(\mathbb{K}) = 0$. □

We call such $\mathbb{K}$ of cardinality $2t+2$ a *star*. By Proposition 4.2, $\alpha(\mathbb{K}) = 0$ for a star $\mathbb{K}$. It is easy to see that convex hull of endpoints of vectors v of a star is a $(2t+1)$-dimensional simplex with origin in its center.

Let $\mathbb{K}$ be a star and let $u \in \mathbb{V} - \mathbb{K}$. Since $vu = \pm 1$ for $v \in \mathbb{K}$, the equality $a(\mathbb{K}) = 0$ implies that $vu = 1$ for one half of vectors v of $\mathbb{K}$ and $vu = -1$ for another half. Hence each vector $u \in \mathbb{V} - \mathbb{K}$ carries out a partition of $\mathbb{K}$ into equal parts. The set of vectors $v \in \mathbb{V} - \mathbb{K}$ carrying out the same partition of $\mathbb{K}$ is called a *pillar* [10].

It follows that the set $\mathbb{V} - \mathbb{K}$ is partitioned into disjoint pillars. Since there are

$$p(t) = \frac{1}{2}\binom{2t+2}{t+1} \tag{4.3}$$

partitions of $\mathbb{K}$, $\mathbb{V}$ contains at most $p(t)$ pillars.

Obviously each pillar $\mathbb{P}$ is a symmetric set because w and $-w$ carry out the same partition.

For any partition $\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2$ with $|\mathbb{K}_i| = t+1$ the vector

$$e = a(\mathbb{K}_1)/(t+1) = -a(\mathbb{K}_2)/(t+1) \tag{4.4}$$

with a from (4.2), has the norm 1. Besides

$$ve = 1 \text{ for } v \in \mathbb{K}_1 \quad \text{and} \quad ve = -1 \text{ for } v \in \mathbb{K}_2.$$

Hence for the vectors $x_v = e - v$, $v \in \mathbb{K}_1$, $y_u = u + e$, $u \in \mathbb{K}_2$, we have

$$x_v^2 = y_v^2 = 2t, \qquad x_v x_{v'} = y_u y_{u'} = -2, \qquad x_v y_u = x_v e = y_u e = 0. \tag{4.5}$$

Hence the vectors x_v, $v \in \mathbb{K}_1$, span a t-dimensional space Q_1 which is orthogonal to the space Q_2, spanned by the vectors y_u, $u \in \mathbb{K}_2$.

5. Sets of equiangular lines with only one filled pillar

Let $\mathbb{V}$ be a symmetric set with only one pillar $\mathbb{P}(\mathbb{K}) = \mathbb{V}$ corresponding to a partition $\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2$. One calls such a set a *pillar set*, i.e. for $w \in \mathbb{P}(\mathbb{K})$ and $v \in \mathbb{K}$ the sign of wv depends only on whether v belongs to $\mathbb{K}_1$ or $\mathbb{K}_2$.

Proposition 5.1. *Let $\mathbb{P}$ be a symmetric $(2t+1, 1)$-system spanning $\mathbb{R}^n$ with a star $\mathbb{K}$. The following are equivalent*
(i) *$\mathbb{P}$ is a pillar set,*
(ii) *there is a vector $e \in \mathbb{R}^n$ of norm 1 such that $ve = \pm 1$ for all $v \in \mathbb{P}$.*

Proof. (i) $\Rightarrow$ (ii): We take e of (4.4). Let $w \in \mathbb{V} - \mathbb{K}$. Then $we = 1$ or -1 according to $wv = 1$ or -1 for all $v \in \mathbb{K}_1$.

(ii) $\Rightarrow$ (i). Since $a(\mathbb{K}) = 0$ for the star $\mathbb{K}$ and a from (4.2), $ea(\mathbb{K}) = 0$, and the vector e partitions $\mathbb{K}$ into equal parts $\mathbb{K}_1 = \{v \in \mathbb{K}\colon ev = 1\}$ and $\mathbb{K}_2 = \{v \in \mathbb{K}\colon ev = -1\}$. Redenote e of (4.4) corresponding to this partition by e_1. We have $ee_1 = 1$. The equality implies $e = e_1$, because e and e_1 have equal norm 1. Hence $we = 1\ (-1)$ for $w \in \mathbb{V} - \mathbb{K}$ implies $wv = 1$ ($wv = -1$) for all $v \in \mathbb{K}_1$. Therefore $\mathbb{P}$ is a pillar set. $\square$

Proposition 5.1 allows one to define a pillar set even if the set has no star. Hence we call a $(2t+1, 1)$-system $\mathbb{P}$ and corresponding set of equiangular lines *pillar* if it satisfies Proposition 5.1(ii). A classical example of a maximal pillar set without a star is $(3, 1)$-system of type A_n in Section 3.

We call the vector e a *shaft* of $\mathbb{P}$. Let

$$\mathbb{P}_+ = \{w \in \mathbb{P}\colon we = 1\}, \qquad x_w = w - e,\ w \in \mathbb{P}_+. \tag{5.1}$$

Note that x_w is a projection of w onto the space orthogonal to e. Since $x_v x_w = vw - 1$ for $v, w \in \mathbb{P}_+$, we have

$$x_w^2 = 2t,\ x_w x_v = 0 \quad \text{iff} \quad wv = 1, \qquad x_w x_u = -2 \quad \text{iff} \quad wv = -1. \tag{5.2}$$

We reformulate a result of [10].

Proposition 5.2. *Let $\mathbb{P}$ be a symmetric $(2t+1, 1)$-system spanning $\mathbb{R}^n$. The following are equivalent*
(i) *$\mathbb{P}$ is a pillar set,*
(ii) *the switching class $\mathscr{G}(\mathbb{P})$ contains a graph G with largest eigenvalue $\lambda_{\max}(G) \leqslant t$.*

Proof. (i) $\Rightarrow$ (ii). The map $w \to x_w$, given in (5.1), transforms the set of vectors of $\mathbb{P}_+$ into a set $\mathbb{S}$ of vectors of the norm $2t$ with two inner products 0 and -2. The Gram matrix of vectors of $\mathbb{S}$ is $2(tI - A)$, where A is the adjacency matrix of the graph $G(\mathbb{P}_+)$. Since the Gram matrix $2(tI - A)$ is positive semidefinite, the maximal eigenvalue of A is not greater than t.

(ii) $\Rightarrow$ (i). Let $\mathbb{P}'$ be a maximal subset of $\mathbb{P}$ without pairs of opposite vectors such that $\lambda_{\max}(G(\mathbb{P}')) \leqslant t$. The Gram matrix of $\mathbb{P}'$ has the form (4.1), where A is the adjacency matrix of the graph $G = G(\mathbb{P}')$. By Proposition 4.1 $G(\mathbb{P}')$ has a $(2t+1, -1, 1)$-representation by vectors $v' = x'_v + e'$. Since a Gram matrix determines vectors up to a unitary transformation U, $x'_v + e' = Uv$. Recall that a unitary transformation preserves the inner product. Hence $v = X_v + e$, where $x_v = U^{-1}x'_v$, $e = U^{-1}e'$. The vector e is a shaft of $\mathbb{P}$, because $ve = (x_v + e)e = e^2 = 1$ for $v \in \mathbb{P}'$. Hence $\mathbb{P}' = \mathbb{P}_+$. $\square$

Proposition 5.2 shows that the graphs of Proposition 4.1 relate to special sets of equiangular lines, namely, to pillar sets.

The set $\mathbb{S} = \{x_v : v \in \mathbb{P}_+\}$ is decomposed into a direct sum of mutually orthogonal nondecomposable sets $\mathbb{S}_i$.

Similarly, the graph $G(\mathbb{S}) \equiv G(\mathbb{P}_+)$ is partitioned into connected components $G(\mathbb{S}_i)$ corresponding to orthogonal components of $\mathbb{S}$. Let $G(\mathbb{P}_+)$ has q components $G_i = G(\mathbb{S}_i)$ with adjacency matrices A_i. The vectors x_w of a component $\mathbb{S}_i$, $1 \leqslant i \leqslant q$, span a space Q_i. The spaces Q_i are mutual orthogonal, because $x_w x_u = 0$ if x_w and x_u belong to different components. Let

$$\mathbb{P}_i = \{v \in \mathbb{P}_+ : x_v \in \mathbb{S}_i\}.$$

Note that $\mathbb{P}_i$ is the set of vertices of the graph $G(\mathbb{S}_i)$, i.e. $G(\mathbb{S}_i) = G(\mathbb{P}_i)$.

By Perron–Frobenius theory the largest eigenvalue of the adjacency matrix A of a connected graph G is simple and the corresponding eigenvector (called *Perron vector* of A and of G) is strictly positive. The Perron vector can be taken integral if the largest eigenvalue is rational.

Lemma 5.3. *Let $G = G(\mathbb{P}')$ be an induced subgraph of $G(\mathbb{P}_+)$ with $\lambda_{\max}(G) = t$, and let A be the adjacency matrix of G. If $z = (z_v : v \in \mathbb{P}')$ is the Perron vector of G, then*

$$\textstyle\sum\{z_v x_v : v \in \mathbb{P}_i\} = 0, \tag{5.3}$$

$$e = \textstyle\sum\{z_v v : v \in \mathbb{P}_i\}/h, \tag{5.4}$$

where e is the shaft of $\mathbb{P}$ and $h = h(G) \equiv \sum\{z_v: v \in \mathbb{P}'\}$. Moreover, G is a connected component of $G(\mathbb{P}_+)$, i.e. $G = G_i = G(\mathbb{P}_i)$ for some $\mathbb{P}_i$, and the set of vectors $\{x_v: v \in \mathbb{P}_i\}$ is a maximal set of vectors with the inner product 0 or -2 and spanning the space Q_i.

Proof. Obviously $2(tI - A)z = 0$ because $Az = tz$. Recall that the matrix $2(tI - A)$ is the Gram matrix of vectors x_v for $v \in \mathbb{P}'$. Hence the (uv) element of the matrix is $x_u x_v$. We have

$$x_u \Sigma_v x_v z_v = \Sigma_v (x_u x_v) z_v = ((2tI - 2A)z)_u = 0 \quad \text{for all } u \in \mathbb{P}'.$$

The equalities imply (5.3), since the vector $\Sigma_v x_v z_v$ lies in the space spanned by the vectors x_u, $u \in \mathbb{P}'$. By (5.1), $x_v = v - e$. Substituting the x_v into (5.3) we obtain (5.4).

Obviously, G is contained in a connected component of $G(\mathbb{P}_+)$. Multiplying (5.3) by x_u we obtain $\Sigma_v\{z_v(x_u x_v): v \in \mathbb{P}'\} = 0$. The last equality implies $x_u x_v = 0$, since $x_u x_v \leqslant 0$ and $z_u > 0$ for all $u \in \mathbb{P}_+ - \mathbb{P}'$, $v \in \mathbb{P}'$, i.e. G is a connected component of $G(\mathbb{P}_+)$.

Suppose that the set $\{x_v: v \in \mathbb{P}'\}$ is not maximal and there is a vector $x \in Q_i$ such that $xx_v = 0$ or -2 for all $v \in \mathbb{P}'$. Let $w = x + e$, then $wv = \pm 1$ for $v \in \mathbb{P}'$, and we can define the graph $G' = G(\mathbb{P}' \cup \{w\})$. The graph G' is connected and contains G_i as an induced subgraph. Hence $\lambda_{\max}(G') > t$, and the matrix $tI - A'$ is not positive semidefinite. This is a contradiction, since $tI - A'$ is the Gram matrix of vectors x_v for $v \in \mathbb{P}' \cup \{w\}$. □

Note that if $G_i = G(\mathbb{P}_i)$ is a component of $G(\mathbb{P}_+)$ with $\lambda_{\max}(G) < t$, then the vectors x_v, $v \in \mathbb{P}_i$, are linearly independent. For the maximal set $\mathbb{P}$ of Section 3 related to the root lattice A_n, $G_i = K_1$, the one-vertex graph, with $\lambda_{\max}(G_i) = 0$ for all i.

Examples of graphs G with $\lambda_{\max}(G) = t$ are regular graphs of degree t, in particular, the complete graph K_{t+1}. The Perron vector of a connected regular graph is the all-one vector j.

For $t = 1$ there is only one graph G with $\lambda_{\max}(G) = 1$, namely K_2.

If $t = 2$, then the graphs G_i with $\lambda_{\max}(G_i) = 2$ are extended Dynkin diagrams of root systems, and the numbers $h(G_i)$ are the Coxeter numbers of the root systems [3, 4]. Hence we call the numbers $h(G_i)$ Coxeter numbers for $t \neq 2$, too.

There are infinitely many graphs G with $\lambda_{\max}(G) = t \geqslant 3$. In general, for $t \geqslant 3$, a classification of all graphs with maximum eigenvalue not greater than t is a difficult problem.

Now we describe the case when $\mathbb{P}$ contains a star.

Proposition 5.4. *Let $\mathbb{P}$ be a pillar set such that $\lambda_{\max}(G(\mathbb{P}_+)) = t$. The following are equivalent:*

(i) *$\mathbb{P}$ contains a star,*

(ii) *$G(\mathbb{P}_+)$ has at least 2 components isomorphic to K_{t+1}.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathbb{K}$ be a star of $\mathbb{P}$, and let e be a shaft of $\mathbb{P}$. The shaft e partitions $\mathbb{K}$ into 2 subsets $\mathbb{K}_\pm = \{v \in \mathbb{K}: ve = \pm 1\}$, where the signs agree. Obviously, $\mathbb{K}_+$ and $-\mathbb{K}_\pm \equiv \{-v: v \in \mathbb{K}_-\}$ belong to $\mathbb{P}_+$, and the graphs $G(\mathbb{K}_+)$ and $G(\mathbb{K}_-)$ are isomorphic to K_{t+1}.

(ii) $\Rightarrow$ (i). Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be subsets of $\mathbb{P}_+$ such that $G(\mathbb{K}_i)$ is isomorphic to K_{t+1}, $i = 1, 2$. It is easy to verify that the set $\mathbb{K}_1 \cup (-\mathbb{K}_2)$ is a star of $\mathbb{P}$. □

Let $d + 1$ be dimension of a pillar set $\mathbb{P}$ with a shaft e. The space spanned by $\mathbb{P}$ is an orthogonal sum of the line spanned by the shaft e and spaces Q_i, $1 \leqslant i \leqslant q$. Let $Q = \sum_1^q Q_i$ and let $\dim Q_i = d_i$. Then $\dim Q = \sum_1^q d_i = d$. We can represent the pillar $\mathbb{P}$ as a packing the space Q by graphs G_i representing sets $\mathbb{S}_i$ spanning the spaces Q_i.

By Lemma 5.3 there is a linear dependency (5.3) between the vectors x_v of a component $\mathbb{S}_i$, if the corresponding graph G_i has $\lambda_{\max}(G_i) = t$. Since t is a simple eigenvalue, the dependency (5.3) is a unique dependency between x_v, $v \in \mathbb{P}_i$. This implies that the endpoints of the vectors x_v are affinely independent, i.e. its convex hull is a simplex S_i. The dimension d_i of the simplex is equal to dimension of the space Q_i spanned by x_v, $v \in \mathbb{P}_i$, and $d_i = \dim Q_i = |\mathbb{P}_i| - 1 = |V(G_i)| - 1$.

If $\lambda_{\max}(G_i) < t$, then x_v, $v \in \mathbb{P}_i$, are linearly independent, and $d_i = |\mathbb{P}_i| = |V(G_i)|$.

Let $\lambda_{\max}(G_i) = t$ for $1 \leqslant i \leqslant p$, and $\lambda_{\max}(G_i) < t$ for $p + 1 \leqslant t \leqslant q$. We can express the number $|\mathbb{P}|$ of lines in a pillar $\mathbb{P}$ as follows. We see above that $|\mathbb{P}_i| = d_i + 1$ for $1 \leqslant i \leqslant p$, and $|\mathbb{P}_i| = d_i$ for $p + 1 \leqslant i \leqslant q$. Since $|\mathbb{P}| = \sum_1^q |\mathbb{P}_i|$ and $d = \sum_1^q d_i$, we have

$$|\mathbb{P}| = d + p. \tag{5.5}$$

Recall that a $(2t + 1, 1)$-system (and corresponding set of equiangular lines) is called extremal if it has maximal cardinality among all $(2t + 1, 1)$-systems (and all sets of equiangular lines) of the same dimension.

Proposition 5.5 (cf. Lemmens and Seidel [10, Theorem 4.2]). *An extremal pillar $(2t + 1, 1)$-system $\mathbb{P}_0$ of dimension $d + 1$ is such that $G(\mathbb{P}_0)$ contains $[d/t]$ graphs K_{t+1} and a graph G with s vertices and $\lambda_{\max}(G) < t$ where $s = d - t[d/t] < t$. The pillar $\mathbb{P}_0$ has $d + [d/t]$ lines.*

Proof. By (5.5) an extremal pillar set has the greatest number p of components G_i with $\lambda_{\max}(G_i) = t$. Hence we shall prove the proposition if we show that the space Q_0 spanned by a complete graph K_{t+1} has minimal dimension $d_0 = t$ among all Q_i spanned by graphs G_i with $\lambda_{\max}(G_i) = t$. Since $d_i = |V(G_i)| - 1$, we need to prove that $|V(G_i)| > |V(K_{t+1})| = t + 1$. But this inequality for G_i with $\lambda_{\max}(G_i) = t$ is implied by the well known inequality $\lambda_{\max}(G) < |V(G)| - 1$, for a graph G distinct from a complete graph [6, p. 21]. □

Since a graph G' with $s < t$ vertices has $\lambda_{\max}(G') \leqslant s - 1 < t - 1$, the graph G of Proposition 5.5 is an arbitrary graph with s vertices. Hence Proposition 5.5 says that there are as many different extremal pillar $(2t+1,1)$-systems of dimension $d+1$ as there are graphs with s vertices.

In particular, for $t=1$, there is only one extremal pillar $(3,1)$-system of type D_n of Section 3. The pillar $(3,1)$-system of type A_n is not extremal.

6. L-polytopes related to pillar sets

Let $\mathbb{P}$ be a $(d+1)$-dimensional pillar set with a shaft e. Since $ve = \pm 1$ for all $v \in \mathbb{P}$, endpoints of vectors $v \in \mathbb{P}$ lie in two affine d-dimensional subspaces $T_{\pm} = \{x: xe = \pm 1\}$. The spaces $T_{\pm}$ are orthogonal to the vector e and contain endpoints of vectors $\pm e$, where the signs agree.

Let $P_{\pm}$ be convex hulls of endpoints of vectors $v \in P_{\pm}$, where $\mathbb{P}_- = -\mathbb{P}_+$. The vertices of the polytopes $\mathbb{P}_+$ and $\mathbb{P}_-$ lie in intersections of the spaces T_+ and T_- with a sphere S^d of squared radius $2t+1$. The polytopes P_+ and P_- are antipodes in the sphere S^d. Hence we describe only P_+. All antipodes mentioned below are considered with respect to this sphere S^d.

We take the endpoint of the shaft e as a new origin. Then the vertices of P_+ are endpoints of vectors x_v, $v \in \mathbb{P}_+$, and the space T_+ is an orthogonal sum of the spaces Q_i, $1 \leqslant i \leqslant q$.

Recall that endpoints of $x_v \in Q_i$ span a simplex S_i. If $\lambda_{\max}(G_i) = t$ for the corresponding graph G_i, then S_i has dimension $d_i = \dim Q_i$, $d_i + 1$ vertices, and contains inside it the origin. Otherwise, S_i has dimension $d_i - 1$ and d_i vertices, and the origin does not belong to S_i.

We consider the case when $q = p$, i.e. when $\lambda_{\max}(G_i) = t$ for all i. In the case P_+ is the convex hull of vertices of mutually orthogonal simplexes S_i with a common center in origin.

Recall that a polytope P is an L-polytope if:

(a) Vertices $v \in V(P)$ of P lie on a sphere of radius r and span the sphere. We suppose that origin is in the center of the sphere.

(b) For all affine combinations $x(z) = \sum_{v \in V(P)} z_v x_v$, with $z_v \in \mathbb{Z}$ and $\sum_v z_v = 1$, of vertices of P we have $x^2(z) \geqslant r^2$, where x_v is a vector describing the vertex $v \in V(P)$.

(c) The equality $x^2(z) = r^2$ is satisfied only if $x(z) = x_v$ for $v \in V(P)$.

Let $V' \subseteq V(P)$. The section of an L-polytope P by an affine subspace spanned by the set V' is an L-polytope, too. Hence, in order to P_+ be an L-polytope it is necessary for all the simplices S_i to be L-polytopes.

Recall that all the simplexes S_i, $1 \leqslant i \leqslant p$, are inscribed into a sphere of squared radius $2t$ (since $x_v^2 = 2t$) with the center in origin. By Lemma 5.3 the center lies inside the simplexes.

Let S_G be a d-dimensional simplex corresponding to a component G. The graph G has $d+1$ vertices $v_0, v_1, \ldots, v_d$. Let $x_k = x_v$ for $v = v_k$, $0 \leqslant k \leqslant d$. Let $(z_0, z_1, \ldots, z_d)$ be the Perron vector of G. Suppose that there is k such that $z_k = 1$. Without loss of generality we can set $k = 0$. Denote the d-vector $(z_1, \dots, z_d)$ by z_G. Since by (5.3) $\sum_{k=0}^{d} z_k x_k = 0$, we have

$$x_0 = -\sum_{1}^{d} z_k x_k. \tag{6.1}$$

Note that vectors $x_0, x_1, \dots, x_d$ form an affine basis of the space spanned by S_G. The set $\mathcal{B} = \{x_1, \dots, x_d\}$ is a linear basis of the space. We introduce two lattices $L_a = L_a(G)$ and $L_0 = L_0(G)$ related to the simplex S_G. The lattice L_a is affinely generated by vertices of S_G. A general point $x(\xi)$ of L_a has the form

$$x(\xi) = \sum_{k=0}^{d} \xi_k x_k, \quad \sum_{k=0}^{d} \xi_k = 1, \quad \xi_k \in \mathbb{Z}. \tag{6.2}$$

The lattice L_0 is linearly generated by the basis $\mathcal{B}$. A general point of L_0 has the form

$$x(\eta) = \sum_{k=1}^{d} \eta_k x_k, \quad \eta_k \in \mathbb{Z}.$$

Recall that $x_k x_l = -2$ if $(v_k v_l)$ is an edge of G, and $x_k x_l = 0$ otherwise. Hence the quadratic form associated with the basis $\mathcal{B}$ is $2\varphi_G$, where

$$\varphi_G(\eta) = \eta^{\mathrm{T}}(tI - A')\eta.$$

Here A' is the adjacency matrix of the graph $G' = G - v_0$, and η is an integral d-dimensional vector. Note that if H is a regular graph of degree t, then $\varphi_H(\eta) = \sum_{(ij) \in E(H')} (\eta_i - \eta_j)^2$.

For two d-vectors $\xi = (\xi_1, \dots, \xi_d)$ and $\eta = (\eta_1, \dots, \eta_d)$ we introduce the inner product $(\xi, \eta) = \sum_1^d \xi_k \eta_k$. In particular, $\xi^2 \equiv (\xi, \xi) = \sum_1^d \xi_k^2$. Let j be all-one d-vector, and $\varepsilon_k = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 stays at the kth place. Note that $j^2 = d$, and $h = h(G) = (j, z_G) + 1$ is the Coxeter number of G, as it was defined in Section 5. Using these notations and (6.1) we rewrite the equalities (6.2) as follows:

$$x(\xi) = \sum_{k=1}^{d} (\xi_k - \xi_0 z_k) x_k, \quad \xi_0 + (\xi, j) = 1. \tag{6.3}$$

Lemma 6.1. *The lattice L_a is a sublattice of L_0 consisting of points $x(\eta) \in L_0$ such that $\eta \in \mathbb{Z}^d$ and $(\eta, j) \equiv 1 \pmod h$.*

Proof. By (6.3) every point of L_a has the form $\sum_1^d \eta_k x_k$ where $\eta_k = \xi_k - \xi_0 z_k$. For this η we have $(\eta, j) = (\xi, j) - \xi_0 (z_G, j)$. Using the second equality of (6.3) and the equality $(z_G, j) = h - z_0 = h - 1$, we obtain $(\eta, j) = 1 - \xi_0 - \xi_0(h-1) = 1 - \xi_0 h \equiv 1 \pmod h$.

Now let $\sum_1^d \eta_k x_k \in L_0$ be such that $(\eta, j) = 1 + ah$, $a \in \mathbb{Z}$. Then we have $\sum \eta_k x_k = \sum (\eta_k - a z_k) x_k + a \sum_{k=1}^d z_k x_k = \sum_1^d \xi_k x_k + \xi_0 x_0$, where $\xi_k = \eta_k - a z_k$, $\xi_0 = -a$, and $\sum_{k=0}^d \xi_k = (\eta - a z_G, j) - a = (\eta, j) - a(h-1) - a = 1$. Hence $\sum \eta_k x_k \in L_a$. $\square$

Proposition 6.2. *The following are equivalent*:
(i) *the simplex* S_G *is an L-polytope of the lattice* L_a *affinely generated by vertices of* S_G,
(ii) *the minimal nonzero value of the quadratic form* $\varphi_G(\eta)$ *for integral* η *with* $(\eta, j) \equiv 1 \pmod h$ *is equal to* t *and* $\varphi_G(\eta) = t$ *only for* $\eta = \varepsilon_k$, $1 \leqslant k \leqslant d$, *and* $\eta = -z_G$.

In other words, the simplex $S_G = \operatorname{conv}\{x_0, x_1, \ldots, x_d\}$ is an L-polytope of the lattice $L_a(G)$ if and only if the vectors $\pm x_i$, $0 \leqslant i \leqslant d$, are only minimal vectors of the lattice $L_0(G)$.

Proof. (i) $\Rightarrow$ (ii): If S_G is an L-simplex of L_a, then $x^2(\xi) \geqslant 2t$, and $x^2(\xi) = 2t$ only for vertices of S_G. Taking in attention (6.3) we have

$$x^2(\xi) = 2\varphi_G(\xi - \xi_0 z_G). \tag{6.4}$$

The vertices v_k and v_0 of S_G are given by coordinates $(\xi_0, \xi) = (0, \varepsilon_k)$ and $(\xi_0, \xi) = (1, 0)$, respectively. Hence corresponding $\eta = \xi - \xi_0 z_G$ is equal to ε_k and $-z_G$, respectively.

(ii) $\Rightarrow$ (i): The proof of Lemma 6.1 shows that if $(\eta, j) \equiv 1 \pmod h$, then $\eta = \xi - \xi_0 z_G$ for some $\xi \in \mathbb{Z}^d$, $\xi_0 \in \mathbb{Z}$. Hence $2\varphi_G(\eta) = x^2(\xi)$, and the only point of L_a with $x^2(\xi) = 2t$ are such that $\xi - \xi_0 z_G = \varepsilon_k$ or $= -z_G$. Multiplying the equalities by j and using the second equality in (6.3), we obtain the equalities for ξ_0: $1 - \xi_0 h = 1$ and $1 - \xi_0 h = -(h-1)$, respectively. Hence $\xi_0 = 0$ or $\xi_0 = 1$. These solutions give $(\xi_0, \xi) = (0, \varepsilon_k)$ and $(\xi_0, \xi) = (1, 0)$, respectively. The corresponding points of L_a are $x(\xi) = x_k$ and $x(\xi) = x_0$. $\square$

Note special cases of graphs when Proposition 6.2 is true.

If $G = K_{t+1}$ is the complete graph, then the lattice L_0 is in proportion to A_t^*, the dual of the root lattice A_t. In fact, in the case the quadratic form $\varphi_G(\eta)$ has the form

$$\varphi_G(\eta) = t \sum_{k=1}^{k} \eta_k^2 - \sum_{k \neq l} \eta_k \eta_l.$$

The form is often reffered to as Voronoi's principal form of the first type. It defines the lattice A_t^* [4, p. 115]. The lattice L_a is the root lattice A_t.

The other case is when $t = 2$, i.e. $\lambda_{\max}(G) = 2$.

Proposition 6.3. *The conditions of Proposition* 6.2 *are true for a simplex* S *corresponding to a connected graph* G *with* $\lambda_{\max}(G) = 2$.

Proof. In this case G is an extended Dynkin diagram. These graphs and corresponding Perron vectors can be found in [3, 4, 10]. The corresponding x_v are roots up to multiple 2. Any point $x(\eta) = \sum_{v \in V} \eta_v x_v$, $\eta_v \in \mathbb{Z}$, is a point of a root lattice L_0. Here V is the set of vertices of G. Minimal vectors of the lattice L_0 are roots. The sphere circumscribing S contains inside only zero point of L_0, and points of L_0 on the sphere are roots. We show that $\sum \eta_v \neq 1 \pmod h$ for the roots distinct from x_v, $v \in V$. Let

$(z_v: v \in V)$ be the Perron vector, and let $u \in V$ be such that $z_u = 1$. (Such u exists always.) Then the set $V - \{u\}$ is a fundamental basis of L_0. Each root of L_0 is represented in the basis with integral coefficients which are either all nonnegative or all nonpositive, and $x_u = -\sum_{v \neq u} z_v x_v$ is the minimal root [2]. In other words if $x(\eta) = \sum \eta_v x_v$ is a root, then either $\eta_v \geqslant 0$ for all v, or $\eta_v \leqslant 0$, and $|\eta_v| \leqslant z_v$. We have $|\sum \eta_v| = \sum |\eta_v| < \sum z_v = h - 1$. Since η_v's are integral and there are at least 2 $\eta_v \neq 0$, this implies that $\sum \eta_v \not\equiv 1 \pmod h$. □

Let $\mathscr{G} = \{G_i: 1 \leqslant i \leqslant p\}$ be the set of graphs of all components. We denote by $V_{p,t}(\mathscr{G})$ the convex hull of vertices of all S_i. The polytope is a special case of the generalized repartitioning polytope $V^{tp}_{t \ldots t}$ of Voronoi (see [15, Section 9]).

Now we show that if the Coxeter numbers h_i are not all equal to each other, then it is possible that $V_{p,t}(\mathscr{G})$ is not an L-polytope.

Consider 2 simplexes S_1 and S_2, and corresponding graphs G_1 and G_2. We show that if $G_i \neq K_{t+1}$, then it is possible that the convex hull $P_{12} = V_{2,t}(\{G_1, G_2\})$ of vertices of S_1 and S_2 is not an L-polytope. Let $V_i = V(G_i)$, $i = 1, 2$, and $|V_1| \leqslant |V_2|$.

Proposition 6.4. *Let $(z_v: v \in V_1)$ be the Perron vector of G_1. Let there be a vertex $u \in V_1$ such that $z_u = 1$ and the graph $G_2 \neq G_1$ has an induced subgraph G'_2 isomorphic to $G_1 \backslash u$. Then P_{12} is not an L-polytope.*

Proof. We show that an endpoint of a vector $x(y) = \sum\{y_v x_v: v \in P_{12}\}$ for some $y_v \in \mathbb{Z}$ such that $\sum y_v = 1$, lies on the sphere S circumscribing the polytope P_{12}, and $x(y)$ is different from a vertex of P_{12}.

Let $V'_2 = V(G'_2)$, $V'_1 = V(G_1 \backslash u)$, and let T be an affine space spanned by x_w for $w \in V_1 \cup V'_2$. For $w \in V'_1$ let w' be a vertex of G'_2 corresponding to w at the above isomorphism.

Recall that the Perron vector $(z_v: v \in V_1)$ is integral. We set

$$y_{w'}, = y_w = z_w \quad \text{for } w \in V'_1, \qquad y_u = z_u = 1. \tag{6.5}$$

Consider the vector

$$x(y) = \sum\{y_w x_w: w \in V_1\} - \sum\{y_{w'} x_{w'}: w' \in V'_2\}. \tag{6.6}$$

Obviously $x(y)$ lies in the space T. Since $y_u = 1$ and $y_w = y_{w'}$, we have $\sum\{y_w: w \in V_1\} - \sum\{y_{w'}: w' \in V'_2\} = 1$. We show that $(x(y))^2 = 2t$. Since, by Lemma 5.3, the first sum in (6.6) is equal to 0, we have

$$(x(y))^2 = \sum\{y_{v'}, y_{w'}, x_{v'}, x_{w'}: v', w' \in V'_2\}.$$

Since the graph G'_2 is isomorph of $G_1 \backslash u$, $x_{v'} x_{w'} = x_v x_w$. Hence using (6.5) we have

$$(x(y))^2 = \sum\{y_v y_w x_v x_w: v, w \in V'_1\} = (\sum\{z_w x_w: w \in V'_1\})^2 = (x_u)^2 = 2t.$$

So, the endpoint of $x(y)$ lies on the sphere S. The vector $x(y)$ does not coincide with x_w for $w \in V_2 - V'_2$, because $G_2 \neq G_1$ and between vectors x_w, $w \in V_2$, there is only one linear dependency given by the Perron vector of G_2. □

Two cases are possible. The first one is when the set of vertices of P_{12}, $V(P_{12})$, is a subset of vertices of some L-polytope. In the case, the distance space $(V(P_{12}), \delta_2)$ is hypermetric for $\delta_2(u, v) = (x_u - x_v)^2$, the squared Euclidean distance.

The second one is when the 2-distance space $(V(P_{12}), \delta_2)$ is not hypermetric, and $V(P_{12})$ belongs to no L-polytope. The proposition below describes such a situation.

Proposition 6.5. *Let h_i be Coxeter numbers of graphs G_i with $\lambda_{\max}(G_i) = t$, $i = 1, 2$. If h_1 and h_2 are mutually prime, then the distance space $(V(P_{12}), \delta_2)$ is not hypermetric.*

Proof. Recall that $r^2 = 2t$ is the squared radius of the sphere circumscribing the polytope P_{12}. If $(V(P_{12}), \delta_2)$ is hypermetric, then for any vector $x = \sum\{y_v x_v: v \in V(P_{12})\}$ with $y_v \in \mathbb{Z}$ and $\sum_v y_v = 1$, we have $x^2 \geqslant 2t$. Since h_1 and h_2 are mutually prime, there are integers p_1 and p_2 such that $p_1 h_1 + p_2 h_2 = 1$. We set $y_v = p_i z_v$ for $v \in V_i$, $i = 1, 2$, where $(z_v: v \in V_i)$ is the Perron vector of the graph G_i, $i = 1, 2$. We have $\sum\{y_v: v \in V_1 \cup V_2\} = p_1 h_1 + p_2 h_2 = 1$. But, since $\sum\{z_v x_v: v \in V_i\} = 0$ for $i = 1, 2$, $x = 0$, too, i.e. x is the center of the circumscribing sphere, and $x^2 < 2t$, a contradiction. □

Proposition 6.5 shows that Conjecture 2.8 is not true in general. Recall that P_{12} is P_+ for a pillar set $\mathbb{P}$ with 2 components. Therefore if we complete the $(2t + 1, 1)$-system $\mathbb{P}$ to a maximal $(2t + 1, 1)$-system $\mathbb{V}$, then the polytope $P(\mathbb{V})$ is not an L-polytope.

Recall the definition of the polytope $V_{p,t}(\mathscr{G})$. $\mathscr{G}$ is a set of graphs G with $\lambda_{\max}(G) = t$, $|\mathscr{G}| = p$. $V_{p,t}(\mathscr{G})$ is the convex hull of vertices of mutual orthogonal simplexes S_G, $G \in \mathscr{G}$. Dimension of $V_{p,t}(\mathscr{G})$ is $\sum_1^p d_i = \sum\{d(G): G \in \mathscr{G}\}$, where $d(G) = |V(G)| - 1$ is dimension of S_G.

Let $P_+ = V_{p,t}(\mathscr{G})$. We denote the convex hull of P_+ and its antipode $P_- = -V_{p,t}(\mathscr{G})$ by $\mathfrak{U}^{p,t}(\mathscr{G})$. Dimension of $\mathfrak{U}^{p,t}(\mathscr{G})$ is equal to $1 + \sum_1^p d_i$. $\mathfrak{U}^{p,t}(\mathscr{G})$ is inscribed into a sphere of squared radius $2t + 1$.

Consider the polytope $\mathfrak{U}^{p,t}(\mathscr{G})$ in detail. Let v_{ik}, $0 \leqslant k \leqslant d_i$, be vertices of the simplex S_i, $1 \leqslant i \leqslant p$. We take the set $\mathscr{B} = \{v_{10}, -v_{10}, v_{ik}, 1 \leqslant k \leqslant d_i, 1 \leqslant i \leqslant p\}$, as the affine basis of the space spanned by $\mathfrak{U}^{p,t}(\mathscr{G})$. Let $(z_{ik}: 0 \leqslant k \leqslant d_i)$ be the Perron vector of G_i, and let $h_i = \sum_k z_{ik}$ be the Coxeter number of G_i. Below we suppose that $z_{i0} = 1$ for all i.

Let $L_{p,t}$ be the lattice affinely generated by the basis $\mathscr{B}$. A point of the lattice $L_{p,t}$ has the form

$$v(\xi) = \xi_{10} v_{10} + \xi_{10}^*(-v_{10}) + \sum\{\xi_{ik} v_{ik}: 1 \leqslant k \leqslant d_i, 1 \leqslant i \leqslant p\}, \tag{6.7}$$

where

$$\xi_{10} + \xi_{10}^* + \sum\{\xi_{ik}: 1 \leqslant k \leqslant d_i, 1 \leqslant i \leqslant p\} = 1. \tag{6.8}$$

Lemma 6.6. *The vertices of $\mathfrak{U}^{p,t}(\mathscr{G})$ belong to $L_{p,t}$ if $h_i = h$ for all graphs $G_i \in \mathscr{G}$.*

Proof. Note that, by (5.4),

$$\sum_{k=0}^{d_i} z_{ik} v_{ik} = h_i e.$$

Hence

$$v_{i0} = (h_i/h_1) \sum_{k=0}^{d_1} z_{1k} v_{1k} - \sum_{k=1}^{d_i} z_{ik} v_{ik}$$

is an integral representation of vertices v_{i0}, $2 \leqslant i \leqslant p$, if $h_i = h_1$. In this case the representation is an affine representation in the basis $\mathscr{B}$, because

$$\sum_{k=0}^{d_1} z_{1k} - \sum_{k=1}^{d_i} z_{ik} = h - (h-1) = 1.$$

Similarly, the vertices of antipodal simplexes $-S_i$ has the following affine integral representation in the basis $\mathscr{B}$: $(-v_{ik}) = v_{10} + (-v_{10}) - v_{ik}$. □

Lemma 6.6 says that the lattice $L_{p,t}$ is affinely generated by the vertices of $\mathfrak{U}^{p,t}(\mathscr{G})$ if $h_i = h$ for all i. Below we show that in this case $\mathfrak{U}^{p,t}(\mathscr{G})$ is an L-polytope of the lattice $L_{p,t}$.

Recall that $v_{ik} = e + x_{ik}$, where $x_{ik} = x_v$ for $v = v_{ik}$. Note that the endpoint of the vector e is the center of all simplexes S_i. Since $\sum_{k=0}^{t} z_{ik} x_{ik} = 0$, we have $x_{i0} = -\sum_{k=1}^{t} z_{ik} x_{ik}$. We set $\gamma = \xi_{10}^* - \xi_{10}$. Using (6.8) we rewrite (6.7) and (6.8) as follows:

$$\begin{aligned} v(\xi) = (1 - 2\xi_{10}^*)e &+ \sum \{\eta_{1k} x_{1k}\colon 1 \leqslant k \leqslant d_i\} \\ &+ \sum \{\xi_{ik} x_{ik}\colon 1 \leqslant k \leqslant d_i,\ 2 \leqslant i \leqslant p\}, \end{aligned} \tag{6.9}$$

$$2\xi_{10}^* - \gamma + \sum_{1}^{p} (\xi_i, j) = 1, \tag{6.10}$$

where $\eta_{1k} = \xi_{1k} + \gamma$. Let $\mathscr{B}_0 = \{e, x_{ik}, 1 \leqslant k \leqslant d_i, 1 \leqslant i \leqslant p\}$ be the linear basis of the space spanned by $\mathfrak{U}^{p,t}(\mathscr{G})$. Let L be the lattice linearly generated by the basis $\mathscr{B}_0$. Any point of L has the form

$$v(s) = s_0 e + \sum \{s_{ik} x_{ik}\colon 1 \leqslant k \leqslant d_i, 1 \leqslant i \leqslant p\}, \quad s_0, s_{ik} \in \mathbb{Z}. \tag{6.11}$$

Recall that the vector e is orthogonal to all x_{ik}, $e^2 = 1$, and $x_{ik} x_{jl} = 0$ if $i \neq j$. Hence the lattice L is the orthogonal sum of the lattice A_1 generated by e and the lattices generated by $\mathscr{B}_i = \{x_{ik}\colon 1 \leqslant k \leqslant t\}$.

The quadratic form associated with L is $v^2(s) = s_0^2 + 2\sum_i \varphi_i(s_i)$, where $\varphi_i = \varphi_G$ for $G = G_i \in \mathscr{G}$ and $s_i = (s_{ik}\colon 1 \leqslant k \leqslant d_i)$.

The lattice $L_{p,t}$ is a sublattice of L_0, since the coefficients s_0, s_{ik} in (6.11) defining $L_{p,t}$ are special.

Lemma 6.7. *The lattice $L_{p,t}$ is the sublattice of L consisting of points $v(s)$ such that s_0 is an odd integer, and $\sum_{i=1}^{p} (s_i, j) = s_0 \pmod h$ where $s_i = (s_{ik}\colon 1 \leqslant k \leqslant d_i)$.*

Proof. By (6.9), each point of $L_{p,t}$ has the form $v(s)$ of (6.11), where $s_0 = 1 - 2\xi_{10}^*$ is an odd integer, and $s_1 = \eta_1 = \xi_1 + \gamma z_1$, $s_i = \xi_i$, $2 \leqslant i \leqslant p$. ($z_1$ is the Perron vector of G_1.) We obtain $(\eta_1, j) = (\xi_1, j) + \gamma(h-1)$. Hence $\sum_1^p (s_i, j) = \sum_1^p (\xi_i, j) + \gamma(h-1)$. Using (6.10) we have $\sum_1^p (s_i, j) = 1 - 2\xi_{10}^* + \gamma h \equiv 1 - 2\xi_{10}^* \pmod h = s_0$.

Now, let $v(s)$ be a point of L such that $s_0 = 1 - 2a$, and $\sum_1^p (s_i, j) = s_0 + \gamma h$, where $\gamma \in \mathbb{Z}$. We set $\xi_i = s_i$, $2 \leqslant i \leqslant p$, and $\xi_1 = s_1 - \gamma z_1$, $\xi_{10}^* = a$, $\xi_{10} = a - \gamma$. Then we have $\sum_1^p (\xi_i, j) = \sum_1^p (s_i, j) - \gamma(z_i, j) = \sum_1^p (s_i, j) - \gamma(h-1) = 1 - 2\xi_{10}^* + \gamma h - \gamma(h-1) = 1 - \xi_{10}^* - \xi_{10}$. Hence the point $v(\xi)$ of (6.7) with just defined ξ determines a point of $L_{p,t}$. □

Proposition 6.8. *Let $h_G = h$ for all $G \in \mathscr{G}$. Let each form φ_G, $G \in \mathscr{G}$, satisfy the following condition. The minimal nonzero value of $\varphi_G(\eta)$ for integral η is equal to t, and for $(\eta, j) = 1 \pmod h$, $\varphi_G(\eta) = t$ only if $\eta = \varepsilon_k$ or $\eta = -z_G$. Then the polytope $\mathfrak{U}^{p,t}(\mathscr{G})$ is an L-polytope of the lattice $L_{p,t}$ generated by its vertices.*

Proof. Recall that $\mathfrak{U}^{p,t}(\mathscr{G})$ is inscribed into a sphere of squared radius $2t+1$. For $\mathfrak{U}^{p,t}(\mathscr{G})$ to be an L-polytope we have to prove that $v^2(\xi) \geqslant 2t+1$ for all points (6.7) of $L_{p,t}$, and $v^2(\xi) = 2t+1$ only for vertices of $\mathfrak{U}^{p,t}(\mathscr{G})$. By Lemma 6.7, each point $v(\xi) \in L_{p,t}$ has the form $v(s)$ of (6.11), where s_0 is an odd integer, and $\sum_i (s_i, j) \equiv s_0 \pmod h$.

Consider the case when $s_i = 0$ for all i. This is possible only if $s_0 = ah$, h is odd, and a is an odd integer. Obviously $h \geqslant t+1$, and $h = t+1$ if $\mathscr{G}$ contains K_{t+1}. Hence $v^2(s) = s_0^2 = a^2h^2 \geqslant a^2(2t+1+t^2) > 2t+1$.

If there is $s_i \neq 0$, then $\varphi_i(s_i) \geqslant t$ ($\varphi_i = \varphi_G$ for $G = G_i$), and $s_0^2 \geqslant 1$. Hence $v^2(s) \geqslant 2t+1$. Therefore the sphere circumscribing $\mathfrak{U}^{p,t}(\mathscr{G})$ is empty. Besides we see, that $v^2(s) = 2t+1$ if and only if $s_0^2 = 1$, and $s_i \neq 0$ only for one i. Let $s_l \neq 0$, $s_i = 0$ for $i \neq l$.

Let $s_0 = 1$. Then $(s_l, j) \equiv 1 \pmod h$. Hence $s_l = \varepsilon_k$ or $s_l = -z_l$. The corresponding $v(s)$ is the vertex v_{lk} or the vertex v_{10} of the simplex S_l. If $s_0 = -1$, then $(s_l, j) \equiv -1 \pmod h$. Then $s_l = -\varepsilon_k$ or $s_l = z_l$. The corresponding point $v(s)$ is either the vertex $-v_{lk}$ or the vertex $-v_{l0}$ of the antipodal simplex $-S_l$. □

Note that the vertices of $\mathfrak{U}^{p,t}(\mathscr{G})$ lie onto two facets, $V_{p,t}(\mathscr{G})$ and its antipode $-V_{p,t}(\mathscr{G})$. Hence we have the following corollary.

Corollary 6.9. *The polytope $V_{p,t}(\mathscr{G})$ is a polytope of the lattice affinely generated by its vertices.*

Consider special cases of polytopes $V_{p,t}(\mathscr{G})$ and $\mathfrak{U}^{p,t}(\mathscr{G})$. If $t = 2$, and $\dim V_{p,t} = 24$, then there are 23 L-polytopes of the type $V_{p,t}(\mathscr{G})$. They are the deep holes of the Leech lattice [4, 5]. Note that shallow holes of the Leech lattice are $V_{p,2}(\mathscr{G})$ for $\mathscr{G}$ containing only ordinary Dynkin diagrams. Besides, all $G \in \mathscr{G}$ have the same Coxeter numbers. Venkov [4, Ch. 18] comes to the equality of Coxeter numbers studying even unimodular 24-dimensional lattices. He proves that all irreducible components of the root system of minimal vectors of an unimodular 24-dimensional lattice have the same Coxeter number. Recall one-to-one correspondence between deep holes of the Leech lattice and even unimodular 24-dimensional lattices.

If all $G \in \mathscr{G}$ are the complete graphs, K_{t+1}, we denote the polytopes $V_{p,t}(\mathscr{G})$ and $\mathfrak{U}^{p,t}(\mathscr{G})$ simply $V_{p,t}$ and $\mathfrak{U}^{p,t}$.

The L-polytope $V_{2,t} = V_{tt}^{2t}$ is the usual repartitioning L-polytope. $\mathfrak{U}^{p,t}$, the convex hull of $P_+ = V_{tt}^{2t}$ and of its antipode P_- is an L-polytope which Baranovskii [1] denotes $\mathfrak{U}^{2t+1}$, and Conway and Sloane [5] call diplosimplex. The polytope is the convex hull of two antipodal regular simplexes of odd dimension $2t + 1$. (Note that the similar convex hull of antipodal simplexes of even dimension is not an L-polytope.) Each of these simplexes is the convex hull of endpoints of a star $\mathbb{K}$ having $2t + 2$ vectors. If we denote the two stars by $\mathbb{K}_i$, $i = 1, 2$, with a partition $\mathbb{K}_i = \mathbb{K}_i^+ \cup \mathbb{K}_i^{-1}$ corresponding to the shaft e, then V_{tt}^{2t} is the convex hull of endpoints of $\mathbb{K}_1^+ \cup \mathbb{K}_2^+$. Note that $-\mathbb{K}_1^+ = \mathbb{K}_2^+$ and $-\mathbb{K}_2^+ = \mathbb{K}_1^-$. It can be shown that in the case the lattice $L_{2,t} = A_{2t+1}^{t+1}$ where A_n^r ($r = t + 1$, $n = 2t + 1$) is the Coxeter lattice.

Note that $V_{p,t}$ is symmetric if and only if $t = 1$. In the case $V_{p,1} \equiv V_{p,1}(K_2) = \operatorname{conv}\{V_{p,1}(K_1) \cup (-V_{p,1}(K_1))\} = \mathfrak{U}^{p,1}(K_1)$. Recall that K_1 and K_2 are the only graphs which can be components of a pillar set for $t = 1$. Therefore, for $t = 1$ we have only 2 types of L-polytopes: $\mathfrak{U}^{p,t}(K_1) = V_{p,1}$ and $\mathfrak{U}^{p,t}(K_2) = \mathfrak{U}^{p,1}$.

The polytope $V_{p,1}$ is the cross polytope $CP(p)$. This L-polytope affinely generates the root lattice D_p. The polytope $\mathfrak{U}^{p,1}$ is the direct product $I \otimes CP(p)$, where I is a segment of norm 4. The endpoints of I are endpoints of vectors e and $-e$. Hence the first 2 columns of Table 1 correspond to sets of equiangular lines with only one pillar.

It can be shown that the construction of Section 2 applied to the Coxeter–Todd lattice of dimension 12 gives the 11-dimensional L-polytope $\mathfrak{U}^{5,2}$.

7. Sets of equiangular lines with several pillars

Let $\mathbb{V}$ be a set of vectors of norm $2t + 1$ spanning a set of equiangular lines at angle $\arccos(2t + 1)^{-1}$, i.e. $\mathbb{V}$ is a $(2t + 1, 1)$-system. Let d be dimension of the space spanned by $\mathbb{V}$. There are two bounds on maximal number of equiangular lines in a d-dimensional space: a special bound $n_s(t, d)$ for $d < (2t + 1)^2$ and an absolute bound $n_a(d)$ not depending on t:

$$n_s(t,d) = d((2t+1)^2 - 1)/((2t+1)^2 - d), \qquad n_a(d) = d(d+1)/2. \tag{7.1}$$

A relation between these bounds is described in [10].

The most important $(2t + 1,1)$-systems correspond to regular two-graphs. We call such a system *regular*, too. It is well known [10] that the special bound is satisfied as an equality for a $(2t + 1, 1)$-system $\mathbb{V}$ if it is regular. In other words, for a regular d-dimensional $(2t + 1, 1)$-system $\mathbb{V}$, we have

$$v(\mathbb{V}) = n_s(t, d).$$

The absolute bound (7.1) says that for any $(2t+1,1)$-system $\mathbb{V}$ of dimension d, $v(\mathbb{V}) \leqslant d(d+1)/2$. Call a regular $(2t+1,1)$-system $\mathbb{V}$, and corresponding two-graph, extreme if it satisfies the absolute bound as an equality, i.e. if $n_s(t,d) = d(d+1)/2$. The equality implies that an extreme $(2t+1,1)$-system has dimension

$$d(t) = (2t+1)^2 - 2$$

(cf. Theorem 3.5 of [10]). Obviously, all regular $(2t+1,1)$-systems have dimension not greater than $d(t)$.

Extreme $(2t+1,1)$-systems are known for $t=1$ and $t=2$. The dimension is 7 and 23, respectively. The extreme systems can be obtained by the construction of Section 1 from the lattice E_8 and the Leech lattice. It is known that corresponding two-graphs are completely regular [11] and unique.

If $\mathbb{V}$ has more than one pillar and $\mathbb{P} \subset \mathbb{V}$ is one of these pillars having shaft e, then each element $u \in \mathbb{V} - \mathbb{P}$ provides a restriction on the graph $G(\mathbb{P}_+)$.

Let G_i be a component of the graph $G(\mathbb{P}_+)$ and let z be its Perron vector. Let $V_i = V(G_i) \subseteq \mathbb{P}_+$ be the set of vertices of G_i. For $u \in \mathbb{V} - \mathbb{P}$ we set

$$h_i^{\pm}(u) = \sum\{z_v\colon v \in V_i,\ vu = \pm 1\}, \quad h_i = h_i^{+}(u) + h_i^{-}(u), \tag{7.2}$$

where the signs agree. Here $h_i = h(G_i)$ is the Coxeter number of G_i. Using (5.4) we obtain

$$ue = (h_i^{+}(u) - h_i^{-}(u))/h_i = 2h_i^{+}(u)/h_i^{-1} = 1 - 2h_i^{-}(u)/h_i. \tag{7.3}$$

The equalities (7.3) imply validity of the following lemma.

Lemma 7.1. *The ratios $h_i^{+}(u)/h_i$ and $h_i^{-}(u)/h_i$ do not depend on i.*

Of course, u belongs to another pillar $\mathbb{P}'$. Let e' be the shaft of the pillar $\mathbb{P}'$. We show that if $\mathbb{V}$ has a star $\mathbb{K}$, then the ratio $h_i^{+}(u)/h_i$ depends, in fact, on the pillar $\mathbb{P}'$ but not on a special $u \in \mathbb{P}'$.

We need the following lemmas.

Lemma 7.2. *Let $u = x_u + e$ and $v = x_v + e$ belong to distinct components of a pillar with the shaft e. Then $ux_v = vx_u = 0$.*

Proof. Note at first that $x_u x_v = 0$, since u and v belong to different components. Hence $ux_v = (x_u + e)x_v = 0$, because $ex_v = 0$. $\square$

Note that if $\mathbb{V}$ has a star $\mathbb{K}$, then $\mathbb{K}$ belongs to any pillar $\mathbb{P}(e) \subseteq \mathbb{V}$. This implies that any shaft e lies in the space $Q(\mathbb{K})$ spanned by $\mathbb{K}$. (Recall that $e = \sum\{v\colon v \in \mathbb{K}, ve = 1\}/(t+1)$). By Lemma 7.2, x_u is orthogonal to the space $Q(\mathbb{K})$. This implies the following lemma.

Lemma 7.3. *Let $\mathbb{V}$ have a star $\mathbb{K}$, and let $\mathbb{P}(e)$ and $\mathbb{P}(e')$ be two pillars with shafts e and e'. Then*

$$x_v e' = x_u e = 0 \quad \textit{for all } v \in \mathbb{P}(e) - \mathbb{K},\ u \in \mathbb{P}(e') - \mathbb{K}.$$

Using (7.3) and Lemma 7.3 we obtain

$$e'e = (u - x_u)e = ue = 2h_i^+(u)/h_i - 1$$

for any u from the pillar $\mathbb{P}(e')$. Recall that a shaft is defined up to sign. We choose e' such that $e'e < 0$, and for this e' we set

$$a(e, e') = h_i^+(u)/h_i = (e'e + 1)/2. \tag{7.4}$$

Lemma 7.1 describes a restriction on possibilities of packing several pillar subsets of $\mathbb{V}$. But the most important restriction is that the cardinality of a pillar set $\mathbb{P}$ of the set $\mathbb{V}$ with several pillars is bounded by amount not depending on dimension of $\mathbb{P}$.

Conjecture 7.4. Let $(2t + 1, 1)$-system $\mathbb{V}$ has more than 1 pillar. Then there is a function $f(t)$ not depending on dimension $\mathbb{V}$ such that $|\mathbb{P}| \leqslant f(t)$ for any pillar $\mathbb{P} \subset \mathbb{V}$.

Conjecture 7.2 is true for $t = 1$ and $t = 2$. The following fact, proved by Neumaier justifies Conjecture 7.4.

Theorem 7.5 (Neumaier [14]). *Let $\mathscr{G}_t$ denote the class of finite graphs with largest eigenvalue $> t$ such that each proper induced subgraph has largest eigenvalue $\leqslant t$. If $\mathscr{G}_t$ is finite, then there is a number $v(t)$ such that every graph with more than $v(t)$ vertices having a $(2t + 1, -1, 1)$-representation is switching equivalent to a graph with largest eigenvalue $\leqslant t$.*

In other words, according to Proposition 5.2, Theorem 7.5 says that if $\mathscr{G}_t$ is finite, then a set of equiangular lines having more than $v(t)$ lines is pillar. Unfortunately, $\mathscr{G}_t$ is not finite for $t \geqslant 3$.

Now we prove a fact (due to [10]) which justifies Conjecture 7.4 once more. The fact says that if a set of equiangular lines has a star and contains more than one pillar, then each pillar contains a bounded number of components, but each component may be arbitrarily large.

Proposition 7.6. *Let $\mathbb{V}$ has a star $\mathbb{K}$ and at least two pillars $\mathbb{P}$ and $\mathbb{P}'$ with shafts e and e', respectively. Then the number s of components of each pillar is bounded,*

$$s \leqslant t^2/a^2(e, e'), \tag{7.5}$$

where $a(e, e')$ is given in (7.4).

Proof. Let x_i be x_v for $v = v_i \in \mathbb{P} - \mathbb{K}$, $1 \leqslant i \leqslant s$. Let $x_i x_j = 0$ for $i \neq j$. For example, this is so if all x_i belong to distinct components G_i of $\mathbb{P}$. Let $u \in \mathbb{P}' - \mathbb{K}$. By Lemma 7.3, $x_i e' = 0$ for all i. Hence $v_i u = (e + x_i)(e' + x_u) = ee' + x_i x_u$, i.e. $x_i x_u = v_i u - ee' = \pm 1 + 1 - 2a(e, e') = 2(b_i - a)$, where we set, for simplicity sake, $a \equiv a(e, e')$, and $b_i = (v_i u + 1)/2 = 0$ or $= 1$. Let b be the s-vector with (0, 1)-components b_i. Recall that $x_i^2 = x_u^2 = 2t$, and j is the all-one vector. The Gram matrix of vectors x_i, $1 \leqslant i \leqslant s$, and x_u is

$$2B = \begin{pmatrix} 2tI_s & 2(b - aj) \\ 2(b - aj)^{\mathrm{T}} & 2t \end{pmatrix}.$$

Since the matrix $2B$ is positive semidefinite, $\det B \geqslant 0$. We use the formula

$$\det \begin{pmatrix} A & a \\ b^{\mathrm{T}} & \beta \end{pmatrix} = (\beta - b^{\mathrm{T}} A^{-1} a) \det A,$$

which can be obtained by expanding

$$\det \begin{pmatrix} A & a \\ b^{\mathrm{T}} & \beta \end{pmatrix}$$

by the last column and the last row. Therefore,

$$\det B = (t - (b - aj)^{\mathrm{T}} (b - aj)/t) t^s \geqslant 0,$$

i.e. $(b - aj)^2 \leqslant t^2$. Note that $b^2 = (b^{\mathrm{T}}, j) = (j^{\mathrm{T}}, b)$, $j^2 = s$, since b is a (0, 1)-vector, and j is an all-one s-vector. Hence $(b - aj)^2 = b^2 - 2a(b^{\mathrm{T}}, j) + a^2 j^2 = b^2(1 - 2a) + a^2 s = b^2 \times (-ee') + a^2 s \leqslant t^2$. Since $b^2 \times (-ee') \geqslant 0$, we obtain $s \leqslant t^2/a$, i.e. (7.5). □

In fact, Proposition 7.6 says that any anticlique of the graph $G(\mathbb{P})$ has size bounded by $t^2/\max_{e'} a^2(e, e')$ for each pillar $\mathbb{P}$ with the shaft e. Since a clique of $G(\mathbb{P})$ is a subgraph of a star, by Proposition 4.2 the size of a clique of $G(\mathbb{P})$ is not greater than $2t + 2$ (cf. Proposition 2.4 of [14]).

8. Taylor graphs

Let $\mathbb{V}$ be an $(M, 1)$-system. The vectors of $\mathbb{V}$ span a set of equiangular lines. Let $\mathbb{V}_1$ be a subset of $\mathbb{V}$ containing exactly one vector from each pair of opposite vectors of $\mathbb{V}$. The vectors of $\mathbb{V}_1$ represent the equiangular lines.

We can relate to the system $\mathbb{V}$ two two-graphs (for definitions of two-graphs see [18, 20]). A triple $\{v_1, v_2, v_3\} \subset \mathbb{V}_1$ belongs to the first two-graph $\mathscr{B}(\mathbb{V})$ if $(v_1 v_2)(v_2 v_3)(v_3 v_1) < 0$ [20]. Just this two-graph is related usually to a set of equiangular lines. The second two-graph $\bar{\mathscr{B}}(\mathbb{V})$ is complementary to $\mathscr{B}(\mathbb{V})$.

Similarly as the set $\mathbb{V}$ corresponds to the two-graph $\mathscr{B}(\mathbb{V})$, there is a set $\mathbb{V}'$ which corresponds to the two-graph $\bar{\mathscr{B}}(\mathbb{V}) = \mathscr{B}(\mathbb{V}')$. Let $\mathscr{G}(\mathbb{V}')$ be the class of switching equivalent graphs related to $\mathbb{V}'$ similarly as $\mathscr{G}(\mathbb{V})$ corresponds to $\mathbb{V}$ (see Section 4). Since $\mathscr{B}(\mathbb{V})$ and $\mathscr{B}(\mathbb{V}')$ are complementary, for each $G \in \mathscr{G}(\mathbb{V})$ there is a graph $G' \in \mathscr{G}(\mathbb{V}')$ such that $G' = \bar{G}$, the complement of G.

Now we construct two graphs $\Gamma_{\pm}$ related to the $(M, 1)$-system $\mathbb{V}$. They are Taylor graphs if the corresponding two-graphs are regular. (For definition of Taylor graph see [3]). The set $\mathbb{V}$ is the set of vertices of both the graphs. A pair (u, v) is an edge of the graph $\Gamma_{\pm} = \Gamma_{\pm}(\mathbb{V})$ if and only if $uv = \pm 1$, and the signs agree.

Note that $\Gamma_- = G + G^*$ and $\Gamma_+ = \bar{G} + \bar{G}^*$ for any $G \in \mathscr{G}(\mathbb{V})$, and $\Gamma_+ \neq \Gamma_-$, since v and v^* are not adjacent in both the graphs. The operation $\gamma(G) = G + G^*$ was introduced in [7, Section 5]. $G + G^*$ is the following graph. G is the restriction of Γ_- on $\mathbb{V}_1$. G^* is an isomorphic copy of G. The operation $* : v \to v^*$ is a bijection between $V(G)$ and $V(G^*)$, the set of vertices of G and G^*. Vertices $u \in V(G)$ and $v^* \in V(G^*)$ are adjacent in $G + G^*$ if and only if $u \neq v$ and (u, v) is not an edge of G.

The following proposition describes properties of $\gamma(G)$ for an arbitrary graph G.

Proposition 8.1. *Let G be a graph with n vertices. Let $A(G)$ and $B = B(G) \equiv J - I - 2A(G)$ be usual and (± 1)-adjacency matrices of G. Then*

(i) *$\gamma(G') = \gamma(G)$ for any graph G' switching equivalent to G,*

(ii) *$\gamma(G)$ is regular of valency $k = n - 1$,*

(iii) *$\gamma(G)$ is connected unless G is the complete graph K_n,*

(iv) *the adjacency matrices of $\gamma(G)$ and $\gamma(\bar{G})$ are*

$$A(\gamma(G)) = \begin{pmatrix} A(G) & A(\bar{G}) \\ A(\bar{G}) & A(G) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} J - I - B & J - I + B \\ J - I + B & J - I - B \end{pmatrix} \equiv g(B), \tag{8.1}$$

$$A(\gamma(\bar{G})) = g(-B), \tag{8.2}$$

(v) *the eigenvalues of $\gamma(G)$ $(\gamma(\bar{G}))$ are k, -1, and minus the eigenvalues of $B(G)$ $(-B(G))$, respectively.*

Proof. Since the properties (i)–(iii) are obvious, we prove (iv) and (v). The first equality in (8.1) is implied by the construction of $\gamma(G)$. The equality $A(\bar{G}) = J - I - A(G)$ implies the second equality of (8.1) and the equality $B(\bar{G}) = -B(G)$. This implies (8.2).

Let $(x, y)^{\mathrm{T}}$ be a vector with the natural partition. Then

$$A(\gamma(G))\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (J - I)\ (x + y) - B(x - y) \\ (J - I)\ (x + y) + B(x - y) \end{pmatrix}.$$

We see that $(x, y)^{\mathrm{T}}$ is an eigenvalue of $A(\gamma(G))$ if and only if either $x = y$ or $x = -y$. In the first case x is an eigenvector of the matrix $J - I$ and eigenvalues of $\gamma(G)$ are the eigenvalues of $J - I$, i.e. they are k and -1. In the second case x is an eigenvector of the matrix B and eigenvalues of $\gamma(G)$ are minus the eigenvalues of $B(G)$. $\square$

The matrix B has exactly two eigenvalues if and only if the two-graph $\mathscr{B}(\mathbb{V})$ is regular [20]. Eigenvalues of B are called eigenvalues of $\mathscr{B}(\mathbb{V})$, too. Since $B(\bar{G}) = -B(G)$, we have that eigenvalues of $\mathscr{B}(\mathbb{V}')$ are minus eigenvalues of $\mathscr{B}(\mathbb{V})$.

Let the graph G corresponds to $\mathbb{V}_1$. Then the matrix $(2t+1)I + B(G)$ is the Gram matrix of the set $\mathbb{V}_1$. Since it is positive semidefinite, $m \leqslant 2t+1$, where $-m$ is the least eigenvalue of $\mathscr{B}(\mathbb{V})$. Suppose that $|\mathbb{V}_1| = v(\mathbb{V}) > \dim \mathbb{V}$. Then the Gram matrix of $\mathbb{V}_1$ is singular, and therefore $m = 2t+1$. Hence the acute angle φ between lines spanned by $\mathbb{V}$ and the least eigenvalue $-m$ of $\mathscr{B}(\mathbb{V})$ are related as $\cos\varphi = 1/m$. Similarly, $\cos\varphi' = 1/n$ for the acute angle φ' between lines spanned by $\mathbb{V}'$ and the least eigenvalues $-n$ of $\mathscr{B}(\mathbb{V}') = \bar{\mathscr{B}}(\mathbb{V})$.

We see that eigenvalues of the graph $\Gamma_\pm$, distinct from k and -1, are eigenvalues of $\mathscr{B}(\mathbb{V})$ multiplied by ± 1, respectively.

Now suppose that the two-graph $\mathscr{B}(\mathbb{V})$ is regular. In the case $\bar{\mathscr{B}}(\mathbb{V})$ is regular, too. Then Γ_+ and Γ_- are distance-regular Taylor graphs of diameter 3 and $\Gamma_+ = (\Gamma_-)_2$, i.e. two vertices adjacent in Γ_+ if and only if the distance between them in Γ_- is equal to 2. In this case Γ_- and Γ_+ have the same parameters with λ and μ interchanged. In other words if $(k, \mu, 1; 1, \mu, k)$ is the intersection array of Γ_-, then $(k, \lambda, 1; 1, \lambda, k)$ is the intersection array of Γ_+. Recall that $\lambda + \mu + 1 = k$.

Each distance-regular graph has a spherical representation corresponding to each eigenvalue of the graph. Recall now the following proposition.

Proposition 8.2 (Brouwer et al. [3, Proposition 4.4.1]). *Let Γ be a distance-regular graph valency k, diameter d, and intersection numbers a_i, b_i, c_i, and let θ be an eigenvalue of Γ of multiplicity f. Then Γ has a spherical representation $p: V(\Gamma) \to \mathbb{R}^f$ such that*

$$p_v p_{v'} = u_i \quad \text{for all } v, v' \in V(\Gamma) \text{ with } d(v, v') = i, \tag{8.3}$$

where $(u_0, u_1, \ldots, u_d)$ is the standard sequence corresponding to θ, i.e.

$$u_0 = 1, \quad u_1 = \theta/k, \quad c_i u_{i-1} + b_i u_i + a_i u_{i+1} = \theta u_i, \qquad 1 \leqslant i \leqslant d,\ |u_i| \leqslant 1. \tag{8.4}$$

Since $u_0 = 1$, (8.3) shows that $p_v^2 = 1$ for all $v \in V(\Gamma)$, and $u_i = \cos\varphi_i$, where φ_i is the angle between p_v and $p_{v'}$ with $d(v, v') = i$.

We apply this proposition to a Taylor graph. Recall that its intersection array is $(k, \mu, 1; 1, \mu, k)$, it has diameter $d = 3$, and

$$b_0 = k, \quad b_1 = \mu, \quad b_2 = 1, \quad b_3 = 0, \quad c_1 = 1, \quad c_2 = \mu, \quad c_3 = k.$$

Since $a_i + b_i + c_i = k$, $1 \leqslant i \leqslant d$, we obtain $\lambda \equiv a_1 = a_2 = k - \mu - 1$, $a_3 = 0$. Hence the system (8.4) takes the form

$$1 + \lambda u_1 + \mu u_2 = \theta u_1, \qquad \mu u_1 + \lambda u_2 + u_3 = \theta u_2, \quad k u_2 = \theta u_3.$$

Summing and subtracting the first 2 equations we obtain

$$(1 \pm u_3) + (\lambda \pm \mu)(u_1 \pm u_2) = \theta(u_1 \pm u_2).$$

Since $u_1 = \theta/k$ and $u_2 = \theta/ku_3$, the equations take the form

$$(1 + (\lambda \pm \mu - \theta)\theta/k)(1 \pm u_3) = 0,$$

where the signs agree. Using the equality $\lambda + \mu + 1 = k$, we rewrite the equations as follows

$$(\theta^2 - (k-1)\theta - k)(1 + u_3) = 0,$$

$$(\theta^2 + (\mu - \lambda)\theta - k)(1 - u_3) = 0.$$

The solutions of the equations are as follows:
(1) $u_3 = -1$, and θ satisfies the equation $\theta^2 + (\mu - \lambda)\theta - k = 0$,
(2) $u_3 = 1$, and θ satisfies the equation $\theta^2 - (k-1)\theta - k = 0$.
The equation $\theta^2 - (k-1)\theta - k = 0$ has the solutions $\theta = k$ and $\theta = -1$, and $\theta = k$ is the largest eigenvalue of Γ. These are just the eigenvalues of the matrix $J - I$.

Consider the case (1). We have $u_0 = -u_3 = 1$, $u_1 = -u_2 = \theta/k$. We see that a pair of opposite vectors corresponds to vertices of Γ at distance 3, since $u_3 = -1$. There is only one acute angle φ between vectors p_v and $\cos\varphi = |\theta|/k$. Call the solutions of the equations $\theta^2 + (\mu - \lambda)\theta - k = 0$ *nontrivial* eigenvalues.

If $\mu \neq \lambda$, then nontrivial eigenvalues are integral, have opposite signs, and the product is equal to $-k$. Following to [3], we denote the nontrivial eigenvalues n and $-m$. So, n is the second largest eigenvalue of Γ, and

$$k = nm, \qquad \mu - \lambda = m - n.$$

Since Γ_2 has μ and λ interchanged, the nontrivial eigenvalues of Γ_2 are m and $-n$, and m is the second largest eigenvalue of Γ_2.

So, a Taylor graph Γ has two spherical representations vector of which span equiangular lines. The two sets of equiangular lines are distinct if $n \neq m$.

Now let $\Gamma = \Gamma_+$, and n and $-m$ be its nontrivial eigenvalues. Recall that n and $-m$ are eigenvalues of $\mathscr{B}(\mathbb{V})$, too. The representation of Γ_+ related to $\theta = n$ spans equiangular lines at angle $\arccos(n/k) = \arccos(1/m)$. This is just the angle between the lines spanned by $\mathbb{V}$ if $v(\mathbb{V}) > \dim \mathbb{V}$.

If $v, v' \in \mathbb{V}$ correspond to adjacent vertices of $\Gamma_+(\mathbb{V})$, then $vv' = 1$ and $p_v p_{v'} = 1/m$, i.e. the sign agrees. Recall that $\Gamma_-(\mathbb{V})$ has eigenvalues m and $-n$. Let p_v be representation of $\Gamma_-(\mathbb{V})$ related to the least eigenvalue $-n$. Then, for $v, v' \in \mathbb{V}$ corresponding to adjacent vertices of Γ_-, $p_v p_{v'} = -1/m$ and $vv' = -1$, and signs agree. Hence we obtain the following proposition.

Proposition 8.3. *Let an $(M, 1)$-system $\mathbb{V}$ (spanning a set of equiangular lines) have $v(\mathbb{V}) > \dim \mathbb{V}$, and let the corresponding two-graph $\mathscr{B}(\mathbb{V})$ be regular with eigenvalues n and $-m$. Then $M = m$, and the spherical representation of the distance-regular graph $\Gamma_+(\mathbb{V})$ related to its second largest eigenvalue n is $\mathbb{V}$ up to a multiple. The same representation is related to the least eigenvalue $-n$ of $\Gamma_-(\mathbb{V})$. The representation of*

$\Gamma_{\pm}(\mathbb{V})$ related to $\mp m$ is, up to sign, an $(n, 1)$-system $\mathbb{V}'$, and it corresponds to the complement of the two-graph $\mathscr{B}(\mathbb{V})$.

Note that the condition $v(\mathbb{V}) > \dim \mathbb{V}$ is essential. For example, the Taylor graph C_6, a cycle of the length 6, as an induced subgraph of the cube Q_3, has a spherical representation by a (3, 1)-system $\mathbb{V}$ with $v(\mathbb{V}) = 3 = \dim \mathbb{V} = \dim Q_3$. But the nontrivial eigenvalues of C_6 are 1 and -2. Similarly, the Taylor graph *Ico*, the icosahedron, as a subgraph of the 6-dimensional Halved cube $Hc(6)$, has a representation by a (3, 1)-system with $v(\mathbb{V}) = 6 = \dim \mathbb{V} = \dim Hc(6)$. But the nontrivial eigenvalues of *Ico* are $\pm 5^{1/2}$ (see [7]).

Note that Q_3 and $Hc(6)$ are hypermetric graphs. Taking in attention Conjecture 2.8 we have the following.

Conjecture 8.4. Let θ be a nontrivial eigenvalue of a Taylor graph G with n vertices. Let f be multiplicity of θ, and let $f < n/2$. Then the representation of G related to θ is hypermetric, i.e. the convex hull of endpoints of vectors of the representation is an L-polytope.

9. Planetary model

Recall that a minimal dependency between vectors of a pillar set is induced by the equality (5.4), i.e. it has the form

$$e = \sum\{z_v v\colon v \in \mathbb{P}_i\}/h_i = \sum\{z_v v\colon v \in \mathbb{P}_j\}/h_j.$$

This means that if we have the set $\mathbb{P}_i \cup \mathbb{P}_j - \{v_0\}$ of vectors spanning equiangular lines, then we can add a new line spanned by v_0 without augmenting dimension of the space where the set of lines lie.

Obviously, the vector v_0 belongs to the space Q_j spanned by $\mathbb{P}_j$. Since $v_0 = x_0 + e$ and x_0 is orthogonal to Q_i, for $i \neq j$, $v_0 v = \pm 1$ for all vectors v of the same pillar. In other words, a maximal (see definition in Section 2) set of equiangular lines with only one pillar does not contain a broken circuit. (In theory of matroids, a minimal dependency is called a *circuit*. A circuit with one vector removed is called a *broken* circuit). Unfortunately, a set $\mathbb{V}$ spanning equiangular lines at angle α with several pillars can contain a broken circuit. The vector which closes a broken circuit can span a line at angle with lines of another pillar distinct from α.

If $G_i = G_j = K_{t+1}$, then the circuit $\mathbb{P}_i \cup (-\mathbb{P}_j)$ is a star with the unique dependency $\sum\{v\colon v \in \mathbb{P}_i \cup (-\mathbb{P}_j)\} = 0$.

Note that sets with several pillars can have another circuits of cardinality greater than $2t + 2$ for $t > 1$. But, it can be verified that for $t = 1$ there are only circuits related to stars.

Let $\mathbb{V}$ be a maximal set such that its pillars have as components complete graphs only. Let $v(\mathbb{V}) \geqslant 2t + 2$, and $\mathbb{V}$ contains a star. We fix a star $\mathbb{K}_0$. The star

$\mathbb{K}_0$ determines

$$p(t) = \binom{2t+2}{t+1}/2$$

pillars. We consider the following sequence of sets $\mathbb{V}_i = \mathbb{V}_i(t)$, $i = 0, 1, 2, \ldots$, $\mathbb{V}_0 = \mathbb{K}_0$. Each set $\mathbb{V}_i(t)$, $i > 0$, has $p(t)$ pillars, and each pillar $\mathbb{P}_{+r}$, $1 \leqslant r \leqslant p(t)$, contains i pairs of opposite vectors, not contained in $\mathbb{K}_0$. So $\mathbb{V}_i$ contains $v(\mathbb{V}_i) = 2t + 2 + p(t)i$ lines. We call a set $\mathbb{V}_i - \mathbb{V}_{i-1}$, $i \geqslant 1$, an *orbit*. An orbit contains one pair of opposite vectors in each pillar, and $v(\mathbb{V}_i - \mathbb{V}_{i-1}) = p(t)$.

We partition the orbits into *orbitals* as follows. The vectors of the intersection of an orbital with a pillar form a component K_s, $1 \leqslant s \leqslant t+1$. We call an orbital *full* if all its component are K_{t+1}.

Note that if $\mathbb{V}$ contains an orbital with a component K_t, then $\mathbb{V}$ is not maximal. In fact, in the case each component K_t can be completed to K_{t+1} without augmenting dimension of $\mathbb{V}$, since K_t and a K_{t+1} of $\mathbb{K}_0$ form a broken circuit. Denote the convex hull of endpoints of vectors of the $(2t+1, 1)$-system $\mathbb{V}_i(t)$ by $\mathfrak{B}_i(t)$. Note that $\mathfrak{B}_0(t) = \mathfrak{U}^{2,t}$.

Recall that for $\mathbb{V}$ corresponding to a regular two-graph the special bound n_s is satisfied as equality, i.e. in the case $v(\mathbb{V}) = n_s(t,d) = 4dt(t+1)/((2t+1)^2 - d)$, and the right-hand side is an integer.

If $d = d_0 \equiv (2t+1)^2 - 2$, then $n_s(t,d) = d(d+1)/2 = n_a(t,d)$, and if $d = (2t+1)^2 - 1$, then $n_s(t,d) > n_a(t,d)$. Hence for $d > (2t+1)^2 - 1$ the bound n_a works only.

Below in Tables 2 and 3 we give values of $n_s(t,d)$ for $t = 1$ and $t = 2$, and for d such that $n_s(t,d)$ is an integer. Besides we give number $\mathcal{O}(t,d)$ of orbitals of the corresponding set $\mathbb{V}$. Note that $\mathcal{O}(t,d) = (n_s(t,d) - (2t+2))/p(t)$.

Tables 2 and 3 show that $n_s(t,d) = (2t+2) + \mathcal{O}(t,d)p(t)$ for all pairs (t,d) (excluding $(t,d) = (2,17)$). The case $(t,d) = (2,17)$ does not correspond to a regular two-graph. For $t = 3$ the expression $(n_s(t,d) - (2t+2))/p(t)$ is integral only for $d = 7, 37, 42$ and 47, when $\mathcal{O}(3,7) = 0$, $\mathcal{O}(3,37) = 4$, $\mathcal{O}(3,42) = 8$, $\mathcal{O}(3,47) = 32$.

Table 2

For $t = 1$, $p(t) = 3$, $d_0(t) = 7$

d	1	3	5	6	7
$n_s(t,d)$	1	4	10	16	28
$\mathcal{O}(1,d)$	–	0	2	4	8

Table 3

For $t = 2$, $p(t) = 10$, $d_0(t) = 23$

d	1	5	10	13	15	17	19	20	21	22	23
$n_s(2,d)$	1	6	16	26	36	51	76	96	126	176	276
$\mathcal{O}(2,d)$	–	0	1	2	3	4, 5	7	9	12	17	27

Consider the function $\varphi(t) = n_s(t, d_0(t)) - (2t + 2)$. We have

$$\varphi(t) = 2(t + 1)^2(4t^2 - 1).$$

Recall that

$$p(t) = \binom{2t+2}{t+1}/2.$$

Therefore we have

$$p(1) = 3, \quad \varphi(1) = 2^3 \times 3 = 2^3 P(1);$$

$$p(2) = 10, \quad \varphi(2) = 2 \times 3^2 \times 15 = 3^3 p(2);$$

$$p(3) = 35, \quad \varphi(3) = 2 \times 4^2 \times 35 = 2^5 p(3);$$

$$p(4) = 126, \quad \varphi(4) = 2 \times 5^2 \times 63 = 5^2 p(4);$$

$$p(5) = 6 \times 7 \times 11, \quad \varphi(5) = 2 \times 6^2 \times 99 = (2^2 \times 3^3/7) p(5).$$

In general, for $t \geqslant 5$, $\varphi(t)$ is not divisible by $p(t)$. In particular, the polytope $\mathfrak{B}_{i(t)}(t)$, where $i(t) = \varphi(t)/p(t)$, represents a completely regular two-graph [11] for $t = 1, 2$, and hypothetically for $t = 3$ and 4.

Now consider some examples. Let $t = 1$. Then $p(t) = 3$. Since in the case $K_{t+1} = K_2$, the sets having components K_1 are not maximal. Hence $\mathbb{V}_i$ are maximal for even i only. There are only 3 maximal sets distinct from $\mathbb{K}$. The sets are shown in Table 1 of Section 3.

$i = 0$, $\mathbb{V}_0 = \mathbb{K}$, $v(\mathbb{V}_0) = 4$. The polytope $\mathfrak{B}_0(1) = \mathfrak{U}^{2,1}$ is a 3-dimensional cube, $L(\mathbb{V}_0) = D_4$.

$i = 2$, $v(\mathbb{V}_2) = 4 + 3 \times 2 = 10$. $\mathfrak{B}_2(1)$ is the 5-dimensional Johnson L-polytope $J(6, 3)$, $L(\mathbb{V}_2) = E_6$.

$i = 4$, $v(\mathbb{V}_4) = 4 + 3 \times 4 = 16$. $\mathfrak{B}_4(1)$ is the 6-dimensional Halved cube $Hc(6)$, $L(\mathbb{V}_4) = E_7$.

$i = 8$, $v(\mathbb{V}_8) = 4 + 3 \times 8 = 28$. $\mathfrak{B}_8(1)$ is the 7-dimensional Gosset polytope, $L(\mathbb{V}_8) = E_8$, $i(1) = \varphi(1)/p(1) = 8$.

We see that the 7-dimensional set $\mathbb{V}_6$ is not maximal. All the polytopes $\mathfrak{B}_i(1)$, $i = 2, 4, 8$, represent completely regular two-graphs [11].

Let $t = 2$, $p(t) = 10$. In the case $K_{t+1} = K_3$, and orbitals of a maximal $\mathbb{V}$ contain K_1 and K_3 only. For $d = 15$, the set $\mathbb{V}_3(2)$ contains one full orbital and corresponds to the complete regular 2-graph on 36 points. The polytope $\mathfrak{B}_3(2)$ is obtained from the 16-dimensional Barness-Wall lattice by the construction of Section 1. It can be shown that the 10-dimensional set $\mathbb{V}_1(2)$ (having 1 orbital) can be obtained from $\mathbb{V}_3(2)$ by choice of a vector in each pillar of $\mathbb{V}_3(2)$. Note that the two-graph represented by $\mathbb{V}_1(2)$ is dual to the two-graph represented by $\mathbb{V}_4(1)$.

For $d = 13$, there are 4 nonisomorphic two-graphs [18]. The sets $\mathbb{V}(2)$ representing the two-graphs contain two orbits, i.e. they are $\mathbb{V}_2(2)$. The most symmetric $\mathbb{V}_2(2)$ contains only two orbitals each consisting from one orbit with components K_1. The example shows that an extremal (5, 1)-system may contain broken circuits. The set $\mathbb{V}_2(2)$ is not contained in $\mathbb{V}_3(2)$.

For $d = 21$ and $d = 22$ we check the sets $\mathbb{V}(2)$ obtained from the Steiner system $S(5, 8, 24)$. The excellent description of the two-graph corresponding to 21-dimensional set $\mathbb{V}_{12}(2)$ is given by Spence [19]. He proves that there are only two types of stars, special or not. If the star $\mathbb{K}_0$ is special, then $\mathbb{V}_{12}(2)$ contains 4 full orbitals. Otherwise, the set $\mathbb{V}_{12}(2)$ contains 2 full orbitals and 6 orbitals consisting each from one orbit. The components of 9 pillars are 2 complete graphs K_3 and 6 K_1. The tenth pillar has 4 graphs K_3 as components.

The 22-dimensional set $\mathbb{V}_{17}(2)$ contains 4 full orbitals and 5 orbitals each containing one orbit. Hence the components of each pillar are 4 graphs K_3 and 5 graphs K_1. The L-polytopes $\mathfrak{B}_{12}(2)$ and $\mathfrak{B}_{17}(2)$ are sections of the L-polytope $\mathfrak{B}_{27}(2)$ mentioned below.

For $d = 23$, the set $\mathbb{V}_{27}(2)$ contains 9 full orbitals and spans the famous unique set of 276 equiangular lines. The corresponding L-polytope $\mathfrak{B}_{27}(2)$ is obtained from the Leech lattice by the construction of Section 1. The explicit description of $\mathfrak{B}_{27}(2)$ is given in [8]. The sets $\mathbb{V}_3(2)$ and $\mathbb{V}_{27}(2)$ correspond to completely regular two-graphs.

The sets $\mathbb{V}_{32}(3)$ and $\mathbb{V}_{25}(4)$ of dimensions 47 and 79 must contain $8 = 32/4$ and $5 = 25/5$ full orbitals, respectively. Each pair of complete graphs K_{t+1} can be taken as $\mathbb{K}_0$.

Unfortunately the L-polytope $\mathfrak{B}_{32}(3)$ corresponding to the hypothetical completely regular two-graphs on 1128 points [11] can not be obtained from even unimodular 48-dimensional extremal lattice by the construction of Section 1. The set of equiangular lines obtained by the construction from any even unimodular 48-dimensional lattice with minimal norm 6 contains only 50 lines. This number can be calculated using a result of Venkov [21].

Nothing is known about the L-polytope $\mathfrak{B}_{25}(4)$ and the corresponding hypothetical completely regular graph on 3160 points.

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